# ON THE VOLUME OF INNER PARALLEL BODIES

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Abstract. Motivated by a conjecture of Matheron, we provide bounds for the volume of the inner parallel body of a convex body $K$ involving the alternating Steiner polynomial of $K$. As a consequence we get that this conjecture is not true since, in fact, we prove it is not possible to bound the volume of the inner parallel body in terms of just the alternating Steiner polynomial itself.

## 1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets in the $n$-dimensional Euclidean space $\mathbb{R}^n$. The subset of $\mathcal{K}^n$ consisting of all convex bodies with non-empty interior is denoted by $\mathcal{K}^n_0$. Let $B_n$ be the $n$-dimensional unit ball. The volume of a set $M \subset \mathbb{R}^n$, i.e., its $n$-dimensional Lebesgue measure, is denoted by $\mathrm{V}(M)$.

For two convex bodies $K \in \mathcal{K}^n$ and $E \in \mathcal{K}^n_0$ and a non-negative real number $\rho$ the *outer parallel body of $K$ (relative to $E$) at distance $\rho$* is the Minkowski sum $K + \rho E$. On the other hand, for $0 \leq \rho \leq \mathrm{r}(K;E)$ the *inner parallel body of $K$ (relative to $E$) at distance $\rho$* is the set

$$K \sim \rho E = \{x \in \mathbb{R}^n : \rho E + x \subset K\},$$

where the *relative inradius* $\mathrm{r}(K;E)$ of $K$ with respect to $E$ is defined by

$$\mathrm{r}(K;E) = \sup\{r : \exists\, x \in \mathbb{R}^n \text{ with } x + r\,E \subset K\}.$$

When $E = B_n$, $\mathrm{r}(K;B_n) = \mathrm{r}(K)$ is the classical inradius (see [2, p. 59]). Clearly if $\rho = 0$ the original body $K$ is obtained. Notice that $K \sim \mathrm{r}(K;E)E$ is the set of (relative) incenters of $K$, usually called *kernel* of $K$ with respect to $E$ and denoted by $\ker(K;E)$. The dimension of $\ker(K;E)$ is strictly less than $n$ (see [2, p. 59]). The inner parallel bodies and their properties were studied mainly by Bol [1], Dinghas [3] (see also [6] and [7]) and later by Sangwine-Yager [11].

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The so called *Minkowski-Steiner formula* (or *relative Steiner formula*) states that the volume of the outer parallel body $K+\rho E$ is a polynomial of degree $n$ in $\rho$,

$$\mathrm{V}(K+\rho E)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\rho^i. \tag{1.1}$$

The coefficients $\mathrm{W}_i(K;E)$ are called the *relative quermassintegrals* of $K$, and they are just a special case of the more general *mixed volumes* for which we refer to [13, s. 5.1] and [5, s. 6.2, 6.3]. In particular, we have $\mathrm{W}_0(K;E)=\mathrm{V}(K)$, $\mathrm{W}_n(K;E)=\mathrm{V}(E)$ and $\mathrm{W}_i(K;E)=\mathrm{W}_{n-i}(E;K)$. If $E=B_n$ the polynomial in the right hand side of (1.1) becomes the classical *Steiner polynomial* [14].

Analogous formulae give the value of the relative $i$-th quermassintegral of the outer parallel body $K+\rho E$, namely

$$\mathrm{W}_i(K+\rho E;E)=\sum_{k=0}^{n-i}\binom{n-i}{k}\mathrm{W}_{i+k}(K;E)\rho^k, \tag{1.2}$$

for $\rho\geq 0$ and $i=0,\dots,n$.

There is however no explicit formula for the volume (quermassintegrals) of the inner parallel body of a convex body $K$. It leads to consider the problem of studying if it is possible to get lower/upper bounds for the volume of the inner parallel body in terms of the quermassintegrals of the original body.

It is easy to check that if $E$ is a *summand* of $K$, i.e., if there exists $L\in\mathcal{K}^n$ such that $K=E+L$, then

$$\mathrm{W}_i(K\sim\rho E;E)=\sum_{k=0}^{n-i}\binom{n-i}{k}\mathrm{W}_{i+k}(K;E)(-\rho)^k, \tag{1.3}$$

for $0\leq\rho\leq 1$ and $i=0,\dots,n$. In [10] Matheron proved that the validity of (1.3) for $0<\rho<1$ and $i=0,\dots,n$ implies that $E$ is a summand of $K$. He conjectured that it was enough to assume (1.3) just for $i=0$, i.e., the case of the volume, and even more:

**Conjecture 1.1** (Matheron, [10])**.** *Let $K\in\mathcal{K}^n$. Then for $0\leq\rho<\mathrm{r}(K;E)$,*

$$\mathrm{V}(K\sim\rho E)\geq\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)(-\rho)^i. \tag{1.4}$$

*The equality holds if and only if $E$ is a summand of $K$.*

The right hand side in (1.4) is usually called the *alternating Steiner polynomial* of $K$. Matheron proved Conjecture 1.1 for $n=2$.

On the other hand, convex bodies can be classified depending on the differentiability of the quermassintegrals as functions of the parameter $\rho$ which defines the in- and outer parallel bodies. These classes are called $\mathcal{R}_p$ classes, $0\leq p\leq n-1$ (see Section 2 for the definition and properties). Here we get bounds for the volume of the inner parallel body of a convex body depending on the class where it lies. These bounds involve the alternating Steiner polynomial:

**Theorem 1.1.** *Let $K \in \mathcal{K}^n$ be a convex body lying in $\mathcal{R}_p$, $0 \leq p \leq n-1$. For every $0 \leq \rho < \mathrm{r}(K;E)$ it holds:*
*i) If $p = n-1$ then*

$$\mathrm{V}(K \sim \rho E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)(-\rho)^i. \tag{1.5}$$

*ii) If $p$ is even, $0 \leq p \leq n-2$ then*

$$\begin{aligned} \mathrm{V}(K \sim \rho E) \geq{}& \sum_{i=0}^{p+2} \binom{n}{i} \mathrm{W}_i(K;E)(-\rho)^i \\ &- \binom{n}{p+2}(n-p-2)\int_0^\rho \frac{(\rho-s)^{p+2}}{\mathrm{r}(K;E)-s}\mathrm{W}_{p+2}(K \sim sE;E)\,ds. \end{aligned} \tag{1.6}$$

*iii) If $p$ is odd, $1 \leq p \leq n-2$ then*

$$\begin{aligned} \mathrm{V}(K \sim \rho E) \leq{}& \sum_{i=0}^{p+2} \binom{n}{i} \mathrm{W}_i(K;E)(-\rho)^i \\ &+ \binom{n}{p+2}(n-p-2)\int_0^\rho \frac{(\rho-s)^{p+2}}{\mathrm{r}(K;E)-s}\mathrm{W}_{p+2}(K \sim sE;E)\,ds. \end{aligned} \tag{1.7}$$

*Equality holds in both inequalities if and only if $K$ is homothetic to an $(n-p-2)$-tangential body of $E$.*

For definition and properties of tangential bodies see Section 2. As a consequence of this theorem we get that there are many sets for which inequality (1.4) does not hold, which proves the non-validity of Matheron's conjecture.

**Corollary 1.1.** *Let $K \in \mathcal{K}^n$, $n$ odd, be a convex body lying in $\mathcal{R}_{n-2}$. Then for $0 \leq \rho < \mathrm{r}(K;E)$*

$$\mathrm{V}(K \sim \rho E) \leq \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)(-\rho)^i. \tag{1.8}$$

*The equality holds if and only if $K \in \mathcal{R}_{n-1}$.*

We also show that for many convex bodies it depends on the parity of the dimension.

**Theorem 1.2.** *Let $K \in \mathcal{K}_0^n$ be a 1-tangential body of $E$. If $n$ is odd then inequality (1.8) holds. If $n$ is even then inequality (1.4) holds. In either case equality holds if and only if $K = E$.*

We remark that the non-validity of the conjecture in the 3-dimensional space and for $E = B_3$ was already mentioned in [12].

The problem of giving upper or lower bounds for the volume of the inner parallel body of a convex body in terms of precisely the alternating Steiner polynomial $\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)(-\rho)^i$ is however hopeless (cf. Remark 4.1), since there exist convex bodies verifying either (1.4) or (1.8), when the parity of the dimension is the contrary to the one in Theorem 1.2.

**Theorem 1.3.** *There exist convex bodies in odd (even) dimension for which inequality* (1.4) *(inequality* (1.8)*) holds.*

The paper is organized as follows. In Section 2 we give some preliminary results on differentiability of quermassintegrals, which are needed for the proof of the theorems. Then, in Section 3 we present the proof of Theorem 1.1, as well as some consequences. Finally Section 4 is devoted to discuss the relation between the volume of the inner parallel body and the alternating Steiner polynomial itself. There Theorem 1.2 and Theorem 1.3 are proved.

## 2. On differentiability of quermassintegrals

From now on $E \in \mathcal{K}_0^n$ will be a fixed convex body (with interior points). For a convex body $K \in \mathcal{K}^n$, the *full system of (relative) parallel bodies* of $K$ is defined by

$$K_\rho := \begin{cases} K \sim (-\rho)E & \text{for } -\mathrm{r}(K;E) \le \rho \le 0, \\ K + \rho E & \text{for } 0 \le \rho < \infty, \end{cases} \tag{2.1}$$

which is a concave family, i.e., it satisfies $(1-\lambda)K_\rho + \lambda K_\sigma \subset K_{(1-\lambda)\rho+\lambda\sigma}$ for $\lambda \in [0,1]$ and $\rho, \sigma \in \bigl[-\mathrm{r}(K;E), \infty\bigr)$, [13, p. 135]. Then, Brunn-Minkowski's theorem for relative quermassintegrals (see e.g. [13, p. 339]) assures that

$$'\mathrm{W}_i(\rho) \ge \mathrm{W}_i'(\rho) \ge (n-i)\mathrm{W}_{i+1}(\rho) \tag{2.2}$$

for $i = 0, \ldots, n-1$, where $'\mathrm{W}_i$ and $\mathrm{W}_i'$ denote, respectively, the left and right derivatives of the function $\mathrm{W}_i(\rho) := \mathrm{W}_i(K_\rho; E)$. It is well known (see e.g. [1, 10]) that the volume is always differentiable and $\mathrm{V}'(\rho) = n\mathrm{W}_1(\rho)$. Moreover, if $\rho \ge 0$ then it is clear from (1.2) that all quermassintegrals are differentiable at $\rho$ (notice that in the case $\rho = 0$ we speak about differentiability from the right) and $\mathrm{W}_i'(\rho) = (n-i)\mathrm{W}_{i+1}(\rho)$. The question arises for which convex bodies equalities hold in (2.2) for the full range $-\mathrm{r}(K;E) \le \rho < \infty$.

**Definition 2.1** ([8])**.** *A convex body $K \in \mathcal{K}^n$ belongs to the class $\mathcal{R}_p$, $0 \le p \le n-1$, if for all $0 \le i \le p$, and for $-\mathrm{r}(K;E) \le \rho < \infty$ it holds*

$$'\mathrm{W}_i(\rho) = \mathrm{W}_i'(\rho) = (n-i)\mathrm{W}_{i+1}(\rho). \tag{2.3}$$

Since $\mathrm{V}'(\rho) = n\mathrm{W}_1(\rho)$ then $\mathcal{R}_0 = \mathcal{K}^n$. Moreover $\mathcal{R}_{i+1} \subset \mathcal{R}_i$ strictly for $i = 0, \ldots, n-2$, see [8]. The problem of determining the convex bodies belonging to the class $\mathcal{R}_p$ was originally posed by Hadwiger [6] in the 3-dimensional case and for $E = B_n$. In [8] the general $n$-dimensional problem is studied. The following result characterizes the class $\mathcal{R}_{n-1}$:

**Lemma 2.1** ([8, Theorem 1.1])**.** *The only sets in* $\mathcal{R}_{n-1}$ *are the outer parallel bodies of* $k$*-dimensional convex bodies, for* $0 \leq k \leq n-1$*.*

On the other hand, a convex body $K \in \mathcal{K}^n$ containing the convex body $E$ is called a *p-tangential body* of $E$, $p \in \{0, \dots, n-1\}$, if each $(n-p-1)$-extreme support plane of $K$ supports $E$ [13, pp. 75–76]. Here a supporting hyperplane is said to be $p$-extreme if its outer normal vector is a $p$-extreme direction, i.e., it cannot be written as a sum $u_1 + \cdots + u_{p+2}$, with $u_i$ linearly independent normal vectors at one and the same boundary point of $K$. For further characterizations and properties of $p$-tangential bodies we refer to [13, Section 2.2].

So a 0-tangential body of $E$ is just the body $E$ itself and each $p$-tangential body of $E$ is also a $q$-tangential body for $p < q \leq n-1$. A 1-tangential body is usually called *cap-body*, and it can be seen as the convex hull of $E$ and countably many points such that the line segment joining any pair of those points intersects $E$. An $(n-1)$-tangential body will be briefly called *tangential body*.

The following theorem shows the close relation existing between inner parallel bodies and tangential bodies.

**Theorem 2.1** (Schneider [13, pp. 136–137])**.** *Let* $K \in \mathcal{K}_0^n$ *and* $-\mathrm{r}(K;E) < \rho < 0$*. Then* $K_\rho$ *is homothetic to* $K$ *if, and only if,* $K$ *is homothetic to a tangential body of* $E$*.*

We will make also use of the following result, which gives a characterization of $n$-dimensional $p$-tangential bodies in terms of the quermassintegrals.

**Theorem 2.2** (Favard [4], [13, p. 367])**.** *Let* $K, E \in \mathcal{K}_0^n$*,* $E \subset K$*, and let* $p \in \{1, \dots, n\}$*. Then* $\mathrm{W}_{p-1}(K;E) = \mathrm{W}_p(K;E)$ *if and only if* $K$ *is an* $(n-p)$*-tangential body of* $E$*. In this case,* $\mathrm{W}_0(K;E) = \mathrm{W}_1(K;E) = \cdots = \mathrm{W}_{n-p}(K;E)$*.*

The following lemma provides an upper bound for the (left) derivative of the $i$-th quermassintegral. Thus jointly with (2.2) we get upper and lower bounds for the derivative of $\mathrm{W}_i$. From now on we write $\mathrm{r} = \mathrm{r}(K;E)$ for the sake of brevity.

**Lemma 2.2.** *Let* $K \in \mathcal{K}_0^n$ *and* $-\mathrm{r} < \rho \leq 0$*. Then for* $i = 0, \dots, n-1$*,*

$$'\mathrm{W}_i(\rho) \leq \frac{n-i}{\mathrm{r}+\rho}\, \mathrm{W}_i(\rho).$$

*Equality holds if and only if* $K$ *is homothetic to a tangential body of* $E$*.*

*Proof.* The inradius of an inner parallel body $K_\rho$, $-\mathrm{r} \leq \rho \leq 0$, is given by $\mathrm{r}(K_\rho;E) = \mathrm{r} - |\rho| = \mathrm{r} + \rho$. Then (see [11, Lemma 2.9])

$$\frac{\mathrm{r}+\rho}{\mathrm{r}}\, K \subset K_\rho. \tag{2.4}$$

Let $h \in [0, \mathrm{r})$ such that $-\mathrm{r} < \rho - h \leq \rho \leq 0$. Analogously we have

$$\frac{\mathrm{r}+\rho-h}{\mathrm{r}+\rho}\,K_\rho = \frac{\mathrm{r}(K_{\rho-h};E)}{\mathrm{r}(K_\rho;E)}\,K_\rho \subset K_{\rho-h}$$

and therefore

$$\left(1-\frac{h}{\mathrm{r}+\rho}\right)^{n-i}\mathrm{W}_i(\rho) \leq \mathrm{W}_i(\rho-h).$$

Using this inequality we can bound the (left) derivative of $\mathrm{W}_i$:

$$\begin{aligned}{}'\mathrm{W}_i(\rho) &= \lim_{h\to 0}\frac{\mathrm{W}_i(\rho)-\mathrm{W}_i(\rho-h)}{h}\\ &\leq \lim_{h\to 0}\frac{\left[1-\left(1-\frac{h}{\mathrm{r}+\rho}\right)^{n-i}\right]\mathrm{W}_i(\rho)}{h} = \frac{n-i}{\mathrm{r}+\rho}\,\mathrm{W}_i(\rho).\end{aligned}$$

It shows the inequality. In order to prove the equality case, we suppose that ${}'\mathrm{W}_i(\rho)(\mathrm{r}+\rho) = (n-i)\mathrm{W}_i(\rho)$, for $i \in \{0,\dots,n-1\}$. Since $K \in \mathcal{K}_0^n$, $\mathrm{W}_i(\rho) > 0$ and we can write

$$\int_\rho^0 \frac{{}'\mathrm{W}_i(t)}{\mathrm{W}_i(t)}\,dt = \int_\rho^0 \frac{n-i}{\mathrm{r}+t}\,dt.$$

Hence, $\log \mathrm{W}_i(0) - \log \mathrm{W}_i(\rho) = (n-i)\big[\log \mathrm{r} - \log(\mathrm{r}+\rho)\big]$, from which we obtain

$$\frac{\mathrm{W}_i(K;E)}{\mathrm{W}_i(K_\rho;E)} = \frac{\mathrm{W}_i(0)}{\mathrm{W}_i(\rho)} = \left(\frac{\mathrm{r}}{\mathrm{r}+\rho}\right)^{n-i},$$

or equivalently,

$$\mathrm{W}_i(K_\rho;E) = \mathrm{W}_i\left(\frac{\mathrm{r}+\rho}{\mathrm{r}}\,K;E\right). \tag{2.5}$$

From (2.4) and (2.5) we get an equality,

$$K_\rho = \frac{\mathrm{r}+\rho}{\mathrm{r}}\,K$$

for all $\rho \in (-\mathrm{r}, 0]$. Theorem 2.1 assures that $K$ is homothetic to a tangential body of $E$.

Conversely, if $K$ is homothetic to a tangential body of $E$, we know from the proof of Theorem 2.1 (see [13, p. 137]) that $K_\rho = (1+\rho/\mathrm{r})K$. Hence $\mathrm{W}_i(\rho) = (1+\rho/\mathrm{r})^{n-i}\mathrm{W}_i(K;E)$ for all $i = 0,\dots,n$ and then

$$\begin{aligned}{}'\mathrm{W}_i(\rho) = \mathrm{W}_i'(\rho) &= (n-i)\frac{(\mathrm{r}+\rho)^{n-i-1}}{\mathrm{r}^{n-i}}\mathrm{W}_i(K;E)\\ &= \frac{n-i}{\mathrm{r}+\rho}\left(\frac{\mathrm{r}+\rho}{\mathrm{r}}\right)^{n-i}\mathrm{W}_i(K;E) = \frac{n-i}{\mathrm{r}+\rho}\mathrm{W}_i(\rho).\end{aligned}$$

It concludes the proof of the lemma. $\square$

## 3. Proof of Theorem 1.1

For $0 \le \rho < \mathrm{r}$ we have $\mathrm{W}_i(K \sim \rho E; E) = \mathrm{W}_i(K_{-\rho}; E) = \mathrm{W}_i(-\rho)$ in our notation, for $K \in \mathcal{K}^n$. We start with proving Theorem 1.1, which is a direct consequence of the following more general result.

**Theorem 3.1.** *Let $K \in \mathcal{K}_0^n$ be a convex body lying in $\mathcal{R}_p$, $0 \le p \le n-2$. For every $0 \le t \le \rho < \mathrm{r}$ it holds:*
*i) If $p$ is even, $0 \le p \le n-2$ then*

$$\begin{aligned}\mathrm{V}(-\rho) \ge & \sum_{i=0}^{p+2} \binom{n}{i} \mathrm{W}_i(-t)(t-\rho)^i \\ & - \binom{n}{p+2}(n-p-2) \int_t^\rho \frac{(\rho-s)^{p+2}}{\mathrm{r}-s} \mathrm{W}_{p+2}(-s)\, ds.\end{aligned}$$

*ii) If $p$ is odd, $1 \le p \le n-2$ then*

$$\begin{aligned}\mathrm{V}(-\rho) \le & \sum_{i=0}^{p+2} \binom{n}{i} \mathrm{W}_i(-t)(t-\rho)^i \\ & + \binom{n}{p+2}(n-p-2) \int_t^\rho \frac{(\rho-s)^{p+2}}{\mathrm{r}-s} \mathrm{W}_{p+2}(-s)\, ds.\end{aligned}$$

*Equality holds in both inequalities if and only if $K$ is homothetic to an $(n-p-2)$-tangential body of $E$.*

Notice that (1.6) and (1.7) in Theorem 1.1 are obtained by replacing $t = 0$ in Theorem 3.1, and equality (1.5) is a direct consequence of Lemma 2.1. Notice also that if $\dim K \le n-1$ then $\mathrm{r}(K; E) = 0$ and hence the result in Theorem 1.1 is trivial.

*Proof of Theorem 3.1.* We fix $0 \le \rho < \mathrm{r}$. For $0 \le t \le \rho$ we define the function

$$\begin{aligned}F(t) = \mathrm{V}(-\rho) - & \sum_{i=0}^{p+2} \binom{n}{i} \mathrm{W}_i(-t)(t-\rho)^i \\ & + (-1)^p \binom{n}{p+2}(n-p-2) \int_t^\rho \frac{(\rho-s)^{p+2}}{\mathrm{r}-s} \mathrm{W}_{p+2}(-s)\, ds.\end{aligned}$$

Since $K \in \mathcal{R}_p$ by definition $\mathrm{W}_i$ is differentiable and $\mathrm{W}_i'(s) = (n-i)\mathrm{W}_{i+1}(s)$, for $i = 0, \dots, p$. Then it is an easy computation to check that the first derivative of $F$ is

$$\begin{aligned}F'(t) = (-1)^p \binom{n}{p+1} (\rho-t)^{p+1} \Bigg[ & (n-p-1)\mathrm{W}_{p+2}(-t) - \mathrm{W}_{p+1}'(-t) \\ & + \frac{n-p-1}{p+2}(\rho-t)\left(\mathrm{W}_{p+2}'(-t) - \frac{n-p-2}{\mathrm{r}-t}\mathrm{W}_{p+2}(-t)\right)\Bigg].\end{aligned}$$

From (2.2) we know that

$$(n-p-1)\mathrm{W}_{p+2}(-t)-\mathrm{W}'_{p+1}(-t)\le 0. \tag{3.1}$$

Since $\mathrm{W}'_{p+2}(-t)\le {}'\mathrm{W}_{p+2}(-t)$, we can apply Lemma 2.2 which assures that

$$\mathrm{W}'_{p+2}(-t)-\frac{n-p-2}{\mathrm{r}-t}\mathrm{W}_{p+2}(-t)\le 0. \tag{3.2}$$

Thus if $p$ is even then $F'(t)\le 0$, whereas for $p$ odd we have $F'(t)\ge 0$. Since clearly $F(\rho)=0$ and $t\le\rho$, we conclude that:

- for $p$ even it holds $F(t)\ge F(\rho)=0$, which proves i);
- for $p$ odd it holds $F(t)\le F(\rho)=0$, which proves ii).

Now we deal with the equality case. We have to show that $F(t)$ is identically zero for every fixed $0\le\rho<\mathrm{r}$ if and only if $K$ is homothetic to an $(n-p-2)$-tangential body of $E$. If $F(t)\equiv 0$ then equality must hold in (3.1) and (3.2) for all $t$ and $\rho$, $0\le t\le\rho<\mathrm{r}$. Since $K\in\mathcal{R}_p$, by definition equality holds in (3.1) if and only if $K\in\mathcal{R}_{p+1}$. On the other hand, by Lemma 2.2 equality holds in (3.2) if and only if $K$ is homothetic to a tangential body of $E$. Since the only tangential bodies in $\mathcal{R}_{p+1}$ are the $(n-p-2)$-tangential bodies (see [8, Theorem 1.3]) we get the result.

Conversely, if $K$ is homothetic to an $(n-p-2)$-tangential body of $E$ then by Lemma 2.2 equality holds in (3.2). On the other hand, we have again that $K_{-t}=(1-t/\mathrm{r})K$ which implies $\mathrm{W}_{p+1}(-t)=(1-t/\mathrm{r})^{n-p-1}\mathrm{W}_{p+1}(K;E)$. Hence

$$\begin{aligned}\mathrm{W}'_{p+1}(-t)&=(n-p-1)\frac{1}{\mathrm{r}}\left(1-\frac{t}{\mathrm{r}}\right)^{n-p-2}\mathrm{W}_{p+1}(K;E)\\&=(n-p-1)\left(1-\frac{t}{\mathrm{r}}\right)^{n-p-2}\mathrm{W}_{p+2}(K;E)=(n-p-1)\mathrm{W}_{p+2}(-t)\end{aligned}$$

since if $K$ is homothetic to an $(n-p-2)$-tangential body of $E$ then $\mathrm{W}_{p+1}(K;E)=\mathrm{r}(K;E)\mathrm{W}_{p+2}(K;E)$ (cf. Theorem 2.2). It concludes the equality case and the proof of the theorem. □

Notice that Theorem 1.1 provides both upper and lower bounds for the volume of the inner parallel body of a convex body $K$ lying in the class $\mathcal{R}_p$, $p=1,\dots,n-2$. Since $K\in\mathcal{R}_p\subset\mathcal{R}_{p-1}$, if $p$ is even, and hence $p-1$ is odd, then

$$\begin{aligned}&\sum_{i=0}^{p+2}\binom{n}{i}\mathrm{W}_i(K;E)(-\rho)^i-\binom{n}{p+2}(n-p-2)\int_0^\rho\frac{(\rho-s)^{p+2}}{\mathrm{r}-s}\mathrm{W}_{p+2}(-s)\,ds\\&\le V(-\rho)\le\\&\sum_{i=0}^{p+1}\binom{n}{i}\mathrm{W}_i(K;E)(-\rho)^i+\binom{n}{p+1}(n-p-1)\int_0^\rho\frac{(\rho-s)^{p+1}}{\mathrm{r}-s}\mathrm{W}_{p+1}(-s)\,ds\end{aligned}$$

and similarly for the case when $p$ is odd.

As mentioned in the introduction, Corollary 1.1 is a direct consequence of Theorem 1.1, since if the dimension $n$ is odd then $p = n-2$ is so, and hence we get (1.8) from (1.7). It shows the non-validity of the conjectured inequality (1.4).

**Remark 3.1.** *Notice that equality case in Corollary 1.1 supports the second conjectured property in Conjecture 1.1; namely that the volume of the inner parallel body $K_{-\rho}$ verifies $\mathrm{V}(-\rho) = \sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)(-\rho)^i$ if and only if $E$ is a summand of $K$ (cf. Lemma 2.1).*

The continuity of the functionals involved in Theorem 3.1 allows to assure that this result as well as Theorem 1.1 are true also for the limit case when $\rho = \mathrm{r}$. Thus we get the following corollary.

**Corollary 3.1.** *Let $K \in \mathcal{K}^n$ be a convex body lying in $\mathcal{R}_p$, $0 \le p \le n-1$.*
*i) If $p = n-1$ then*

$$\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)(-\mathrm{r})^i = 0.$$

*ii) If $p$ is even, $0 \le p \le n-2$ then*

$$\sum_{i=0}^{p+2}\binom{n}{i}\mathrm{W}_i(K;E)(-\mathrm{r})^i - \binom{n}{p+2}(n-p-2)\int_0^{\mathrm{r}}(\mathrm{r}-s)^{p+1}\mathrm{W}_{p+2}(-s)\,ds \le 0.$$

*iii) If $p$ is odd, $1 \le p \le n-2$ then*

$$\sum_{i=0}^{p+2}\binom{n}{i}\mathrm{W}_i(K;E)(-\mathrm{r})^i + \binom{n}{p+2}(n-p-2)\int_0^{\mathrm{r}}(\mathrm{r}-s)^{p+1}\mathrm{W}_{p+2}(-s)\,ds \ge 0.$$

*Equality holds in both inequalities if and only if $K$ is homothetic to an $(n-p-2)$-tangential body of $E$.*

**Remark 3.2.** *Notice that in the case when $K \in \mathcal{R}_{n-1}$, i.e., when $K$ is an outer parallel body of some convex body, then the inradius $\mathrm{r} = \mathrm{r}(K;E)$ is a root of the alternating Steiner polynomial.*

In particular, for any convex body $K \in \mathcal{K}^n = \mathcal{R}_0$ we get the following inequalities.

**Corollary 3.2.** *For any convex body $K \in \mathcal{K}^n$ it holds*

$$\begin{aligned}\mathrm{V}(K) \le n\mathrm{W}_1(K;E)\,\mathrm{r} - \frac{n(n-1)}{2}\mathrm{W}_2(K;E)\,\mathrm{r}^2 \\ + \frac{n(n-1)(n-2)}{2}\int_0^{\mathrm{r}}(\mathrm{r}-s)\mathrm{W}_2(-s)\,ds.\end{aligned}$$

*Equality holds if and only if $K$ is homothetic to an $(n-2)$-tangential body of $E$.*

**Corollary 3.3.** *For any convex body $K \in \mathcal{K}^n$ it holds*

$$(n-3)\mathrm{V}(K) + 2\mathrm{W}_1(K;E)\,\mathrm{r} - (n-1)\mathrm{W}_2(K;E)\,\mathrm{r}^2 \geq 0.$$

*Equality holds if and only if $K$ is homothetic to an $(n-2)$-tangential body of $E$.*

*Proof.* Since, up to translations, $\mathrm{r}(K_{-s};E)E \subset K_{-s}$, the monotonicity of the mixed volumes (cf. e.g. [5, p. 97]) implies that $(\mathrm{r}-s)\mathrm{W}_2(-s) \leq \mathrm{W}_1(-s)$. Therefore

$$\int_0^{\mathrm{r}} (\mathrm{r}-s)\mathrm{W}_2(-s)\,ds \leq \int_0^{\mathrm{r}} \mathrm{W}_1(-s)\,ds = \frac{1}{n}\mathrm{V}(K),$$

since the volume is differentiable and $\mathrm{V}'(-s) = -n\mathrm{W}_1(-s)$. Hence, using Corollary 3.2 we get

$$\mathrm{V}(K) \leq n\mathrm{W}_1(K;E)\,\mathrm{r} - \frac{n(n-1)}{2}\mathrm{W}_2(K;E)\,\mathrm{r}^2 + \frac{(n-1)(n-2)}{2}\mathrm{V}(K).$$

Simplifying we get the required inequality. Notice that $(\mathrm{r}-s)\mathrm{W}_2(-s) = \mathrm{W}_1(-s)$ if and only if $K_{-s}$ (and hence $K$) is homothetic to an $(n-2)$-tangential body of $E$ (cf. Theorem 2.2). It concludes the proof. □

**Remark 3.3.** *The* relative circumradius $\mathrm{R}(K;E)$ *of $K \in \mathcal{K}^n_0$ with respect to $E$ is defined as* $\mathrm{R}(K;E) = \min\{R : \exists\, x \in \mathbb{R}^n$ *with* $K \subset x + R\,E\}$*. Notice that* $\mathrm{r}(K;E)\,\mathrm{R}(E;K) = 1$*. Then, using the relation* $\mathrm{W}_i(K;E) = \mathrm{W}_{n-i}(E;K)$*, the inequality of Corollary 3.3 can be rewritten as (we interchange $E$ by $K$ in order to write it with the usual notation)*

$$(n-3)\mathrm{R}(K;E)^2\mathrm{V}(E) + 2\mathrm{R}(K;E)\mathrm{W}_{n-1}(K;E) - (n-1)\mathrm{W}_{n-2}(K;E) \geq 0.$$

*Analogously all the previous inequalities can be rewritten in terms of the relative circumradius.*

## 4. The volume of the inner parallel body and the alternating Steiner polynomial

For the sake of brevity we write $f_{K,E}(\rho) = \sum_{i=0}^n \binom{n}{i}\mathrm{W}_i(K;E)(-\rho)^i$ to denote the alternating Steiner polynomial of $K \in \mathcal{K}^n$ with respect to the fixed convex body $E \in \mathcal{K}^n_0$. Now we prove Theorem 1.2.

*Proof of Theorem 1.2.* If $K \in \mathcal{K}^n_0$ is a 1-tangential body of $E$, Theorem 2.2 asserts that $\mathrm{W}_0(K;E) = \mathrm{W}_i(K;E)$, for all $i = 1, \ldots, n-1$, and so we can rewrite the alternating Steiner polynomial $f_{K,E}(\rho)$ in the following way

$$\begin{aligned} f_{K,E}(\rho) &= \mathrm{V}(K)\left[\sum_{i=0}^{n-1}\binom{n}{i}(-\rho)^i + \frac{\mathrm{V}(E)}{\mathrm{V}(K)}(-\rho)^n\right] \\ &= \mathrm{V}(K)\Big[(1-\rho)^n - \big(1-\alpha(K)\big)(-\rho)^n\Big], \end{aligned}$$

where $\alpha(K)=\mathrm{V}(E)/\mathrm{V}(K)$. Observe that $0<\alpha(K)\leq 1$ since $E\subset K$. On the other hand, since $K$ is a tangential body of $E$ we have that $\mathrm{r}(K;E)=1$ and thus $K_{-\rho}=(1-\rho)K$. Hence $\mathrm{V}(-\rho)=(1-\rho)^n\mathrm{V}(K)$. Therefore

$$\mathrm{V}(-\rho)-f_{K,E}(\rho)=\bigl(1-\alpha(K)\bigr)(-\rho)^n.$$

Clearly if the dimension $n$ is odd (even) the above difference is negative (positive) and inequality (1.8) (inequality (1.4)) holds. Equality holds if and only if $\alpha(K)=\mathrm{V}(E)/\mathrm{V}(K)=1$, i.e., only when $K=E$. □

It is also possible to find examples in odd (even) dimension for which inequality (1.4) (inequality (1.8)) holds.

*Proof of Theorem 1.3.* Let $K\in\mathcal{K}_0^n$ be a 2-tangential body of $B_n$. Then on account of Theorem 2.2 we can rewrite the alternating Steiner polynomial of $K$ as

$$\begin{aligned} f_{K,B_n}(\rho)&=\mathrm{V}(K)\left[\sum_{i=0}^{n-2}\binom{n}{i}(-\rho)^i+n\frac{\mathrm{W}_{n-1}(K)}{\mathrm{V}(K)}(-\rho)^{n-1}+\frac{\mathrm{V}(B_n)}{\mathrm{V}(K)}(-\rho)^n\right]\\ &=\mathrm{V}(K)\Bigl[(1-\rho)^n-n\bigl(1-\beta(K)\bigr)(-\rho)^{n-1}-\bigl(1-\alpha(K)\bigr)(-\rho)^n\Bigr]\end{aligned}$$

where $\beta(K)=\mathrm{W}_{n-1}(K)/\mathrm{V}(K)$ and $\alpha(K)=\mathrm{V}(B_n)/\mathrm{V}(K)$ and where we write for short $\mathrm{W}_i(K)=\mathrm{W}_i(K;B_n)$ as usual in the literature. Notice that $\alpha(K)\leq\beta(K)\leq 1$ (see e.g. [13, p. 367]). Since $K\sim\rho B_n=(1-\rho)K$ we can write the difference $\mathrm{V}(K\sim\rho B_n)-f_{K,B_n}(\rho)$ as

$$\mathrm{V}(K\sim\rho B_n)-f_{K,B_n}(\rho)=\mathrm{V}(K)(-\rho)^{n-1}\Bigl[n\bigl(1-\beta(K)\bigr)-\bigl(1-\alpha(K)\bigr)\rho\Bigr]. \tag{4.1}$$

For the sake of brevity we write $G(\rho,K)=n\bigl(1-\beta(K)\bigr)-\bigl(1-\alpha(K)\bigr)\rho$. Since $0\leq\rho\leq 1=\mathrm{r}(K)$ then $G(\rho,K)\geq G(1,K)$. Hence, if we construct a 2-tangential body $K\in\mathcal{K}_0^n$ of $B_n$ such that $G(1,K)\geq 0$ for any value of the dimension $n$, then we get the desired example: when $n$ is odd (even) the above difference in (4.1) is positive (negative) and therefore inequality (1.4) (inequality (1.8)) holds.

In order to get such a body it is enough to consider the 2-tangential body $K_\lambda$ constructed in [9, Proof of Theorem 1.2], i.e., the convex hull of $B_n$ and 5 points suitably chosen depending on a parameter $\lambda$. In [9] it is shown that $\mathrm{V}(K_\lambda)\geq c_n\lambda^2$ and $\mathrm{W}_{n-1}(K_\lambda)\leq 2\lambda\mathrm{V}(B_n)$, where $c_n$ is a constant depending only on the dimension. Notice that the inequality $G(1,K_\lambda)\geq 0$ is equivalent to the relation $(n-1)\mathrm{V}(K_\lambda)\geq n\mathrm{W}_{n-1}(K_\lambda)-\mathrm{V}(B_n)$, which clearly holds if $\lambda$ is large enough. □

**Remark 4.1.** *Corollary 1.1, Theorem 1.2 and Theorem 1.3 show that in general it is hopeless to give upper or lower bounds for the volume of the inner parallel body of a convex body in terms of exactly the alternating Steiner polynomial. It is necessary to deal with particular families of sets (cf. Corollary 1.1 and Theorem 1.2).*

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