UNIVERSIDAD DE MURCIA

Departamento de Matemáticas

DOCTORAL THESIS

FROM BRUNN-MINKOWSKI TYPE INEQUALITIES TO ROOTS OF GEOMETRIC POLYNOMIALS

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2014

Contents

Preface

Brunn–Minkowski's theory has been recognized as the heart of the classical Geometry of Convex Bodies. Its origin dates from the turn of the 19-th century, mainly due to the works in dimensions two and three of H. Brunn -especially to his Ph.D Thesis in 1887- and H. Minkowski (who is considered the father of this branch of Convex Geometry because of his important contributions). A big part of their results was soon generalized to higher dimensional spaces. Furthermore, some topics that Minkowski just touched briefly, have been deeply studied later and significantly extended.

Trying to make a simple overview of Brunn–Minkowski's theory, one could say that it is just the result of combining two elementary notions for sets in Euclidean space: the *Minkowski addition* (i.e., the vectorial addition), $+$, and the *volume* (Lebesgue measure), $\mathrm{vol}(\cdot)$. The Minkowski sum of convex bodies (compact and convex sets), when combined with the volume, leads to the fundamental *Brunn–Minkowski inequality* on one hand and to both the *Steiner polynomial* and the notion of *mixed volumes* on the other hand.

Regarding the Brunn–Minkowski inequality, its rather simple statement might make it to go unnoticed: it ensures the concavity of the n-th root of the volume functional, i.e., given convex bodies K, E then

$$\mathrm{vol}\big((1-\lambda)K+\lambda E\big)^{1/n} \geq (1-\lambda)\mathrm{vol}(K)^{1/n}+\lambda\mathrm{vol}(E)^{1/n}. \qquad (*)$$

Nevertheless, it would be not possible to collect here the potent extensions of it, some of them very recent, as well as their impact on mathematics and beyond. For instance, this inequality implies, among others, the classical isoperimetric inequality -all mathematicians are aware of this relevant result in the plane- for convex bodies (and other important classes of sets) not only in the plane but in $\mathbb{R}^n$. Moreover, Brunn-Minkowski's inequality is the starting point for a fruitful theory of geometric (and analytic) inequalities.

When computing the volume of the Minkowski sum of K with a homothetic copy with factor $\lambda \geq 0$ of the Euclidean unit ball B_n, namely, the so-called *outer parallel body of K at distance λ*, one gets a polynomial expression in λ of degree n; this is the content of the *Steiner formula*. Regarding the coefficients, two different normalizations are considered, one of them involving the the so-called *quermassintegrals* of K, $\mathrm{W}_i(K)$, whereas the second one uses its *intrinsic volumes*, $\mathrm{V}_i(K)$:

$$\mathrm{vol}(K+\lambda B_n)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K)\lambda^i=\sum_{i=0}^{n}\mathrm{vol}(B_i)\mathrm{V}_{n-i}(K)\lambda^i.$$

Quermassintegrals are relevant functionals associated to the given convex body K and, among them, well-known magnitudes such as the volume or the surface area can be found. In 1975, McMullen normalized the quermassintegrals in the following way:

$$\mathrm{V}_i(K) = \binom{n}{i}\frac{\mathrm{W}_{n-i}(K)}{\kappa_{n-i}};$$

further, he proposed to call these functionals intrinsic volumes because of the fact that they do not depend on the dimension of the embedding space, on the one hand, and that $\mathrm{V}_k(K)$ is the usual k-dimensional volume of K, provided that K is k-dimensional, on the other hand.

An analogous result is obtained in the more general context of the so-called *Minkowski Relative Geometry*, i.e., when the Euclidean unit ball B_n is replaced by an arbitrary convex body E. In this case the above notions of outer parallel body and quermassintegrals may be now rewritten relative to the fixed (so-called *gauge*) body E and, in particular, the relative Steiner formula provides the volume of the Minkowski addition $K + \lambda E$:

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)\lambda^i; \tag{†}$$

the functionals $\mathrm{W}_i(K; E)$ are called *relative quermassintegrals* of K with respect to E. The right-hand side in (†), when considered as a formal polynomial in a complex variable $z \in \mathbb{C}$, is called the (relative) *Steiner polynomial* of K with respect to E, and is denoted by $f_{K;E}(z)$:

$$f_{K;E}(z) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)z^i.$$

These two results involving the volume and the Minkowski addition allow to pose three questions, in principle of different nature but deep down closely related, as we will show throughout the work gathered in this dissertation:

i) As we have previously mentioned, Brunn-Minkowski's inequality can be summarized by stating that the volume is $(1/n)$-concave. Moreover it is well-known that this exponent is necessary in the sense that the volume itself is not a concave function; a fact that could be understood somehow as the natural correction that we must impose to the volume -since it is homogeneous of degree n- in order to obtain such an inequality. Nevertheless, it is not so far away from being it, as it shows the following result found in the literature: if K, E are convex bodies such that there exists a hyperplane H with $K|H = E|H$ (here $|$ denotes the orthogonal projection) then, for all $\lambda \in [0, 1]$,

$$\mathrm{vol}\big((1-\lambda)K + \lambda E\big) \geq (1-\lambda)\mathrm{vol}(K) + \lambda\mathrm{vol}(E). \tag{‡}$$

At this point it is a natural question whether an analogous result can be obtained if a *common projection onto an $(n-k)$-dimensional plane* is assumed, i.e., whether the volume is $(1/k)$-concave under such an assumption.

ii) Brunn-Minkowski's inequality $(*)$ holds with equality, for some $\lambda \in (0,1)$, if and only if K and E either lie in parallel hyperplanes or are homothetic. Regarding the above linear version of Brunn-Minkowski's inequality, good candidates for (pairs of) convex bodies characterizing its equality case are the *sausages* (the pair K, E is a sausage if one of them is the Minkowski addition of the other body and a segment) since, by (†), it is easy to check that for these bodies we have

$$\mathrm{vol}\big((1-\lambda)K + \lambda E\big) = (1-\lambda)\mathrm{vol}(K) + \lambda \mathrm{vol}(E) \tag{§}$$

for any $\lambda \in [0,1]$. Moreover, such a pair of convex bodies satisfies the common projection assumption, and thus one might think that this family allows to characterize the equality case in (‡). Furthermore, at this point it is a natural question whether sausages are the *only* bodies that ensure *linearity of the volume*, i.e., those bodies for which (§) holds for all $\lambda \in [0,1]$. If the answer is negative, would it be possible to get such a characterization under some additional hypothesis? Maybe to assume a fixed (relative) inradius? These questions are closely related to some conjectures relative to the behavior of the Steiner polynomial with respect to summands of convex bodies.

iii) Regarding the Steiner polynomial $f_{K;E}(z)$ and the consequent natural problem of studying properties of its roots, which has been recently treated in the literature, the following questions arise: may the known properties of the roots of the Steiner polynomial be extended to the ones of the *Wills polynomial* $\sum_{i=0}^{n} \mathrm{V}_i(K)z^i$? If it is so, can we deduce similar properties for some general family of geometric polynomials of convex bodies (extending both the Steiner polynomial and the Wills polynomial)? In any case, it would be more gratifying if these general geometric polynomials could arise from a quite natural extension of functionals associated to convex bodies. Moreover, it would be interesting to know whether there exists a relation between the roots of the Steiner polynomial and the Wills polynomial.

We can say, roughly speaking, that this dissertation is devoted, on the one hand, to the study of Brunn-Minkowski's type inequalities, especially when working with projections/sections assumptions and, on the other hand, to the study of the roots of geometric polynomials which arise from a generalization of the so-called *Wills functional*. In the middle, we would find sausages, which turn out to be, up to degenerated convex bodies, the family of 'extremal sets' in relation to some linear improvements of inequalities such as Brunn-Minkowski's inequality or Minkowski's first inequality (and thus also the isoperimetric inequality and Urysohn's inequality); furthermore, this family of convex bodies is strongly connected to some problems relative to the Steiner polynomial.

This work starts with an introductory *first chapter* where we establish the notation and introduce the concepts and results that will be needed later on, both about general Convexity and, in particular, about mixed volumes and other functionals. Thus, in a first section, the important notions such as Minkowski addition, convex body, concave/convex function... are recalled. Next, mixed volumes, quermassintegrals and intrinsic volumes are introduced and we use a part of this section to recall the Wills functional as well as to provide a context for it and collect some integral relations that will be needed further on. Then, we collect, on the one hand, other relevant functionals such as the inradius and the successive minima and, on the other hand, some special convex bodies that will be considered throughout this dissertation such as unit p-balls, sausages or p-tangential bodies. Then, some important inequalities are recalled, most of them relating mixed volumes, such as Brunn-Minkowski's inequality, Minkowski's inequalities, Alexandrov-Fenchel inequality, the isoperimetric inequality... Finally, some well-known results and properties on real polynomials are collected.

The *second chapter* is devoted to the study of refinements of the Brunn-Minkowski inequality, in the sense of 'enhancing' the exponent $1/n$, when assuming that the bodies share a common projection onto an $(n-k)$-plane on the one hand, and for particular families of bodies on the other hand. In the first case, we will show that the expected result of concavity for the k-th root of the volume (cf. (‡)) is not true, i.e., that there exist convex bodies K, E so that $K|H = E|H$ for some $(n-k)$-plane H (and certain $1 < k \leq n$) whereas

$$\mathrm{vol}\big((1-\lambda)K + \lambda E\big)^{1/k} < (1-\lambda)\mathrm{vol}(K)^{1/k} + \lambda \mathrm{vol}(E)^{1/k}$$

holds for all $\lambda \in [0,1]$. Nevertheless, other Brunn-Minkowski type inequalities can be obtained under an $(n-k)$-projection hypothesis (respectively, under a common maximal volume $(n-k)$-section assumption).

In the second case, we will work with the family of *p-tangential bodies*; roughly speaking, these sets are characterized to be those convex bodies which arise in a natural way when classifying, in terms of a parameter p, the (type of) singularities of their boundary points (for the precise definition see Chapter 1). There, we show that for the family of p-tangential bodies, the exponent in Brunn-Minkowski's inequality can be replaced by $1/p$.

In the *third chapter*, we deal with the equality case in (‡), i.e., we prove that under the sole assumption that K and E have an equal volume projection (or a common maximal volume section through parallel hyperplanes to a given one), if equality holds in (‡) for just one value λ in $(0,1)$, then either K, E lie in parallel hyperplanes or the pair K, E is a sausage. However, even having equality for all $\lambda \in [0,1]$, which will be briefly referred as *linearity of the volume*, if no extra assumption on K, E is done, we will show that such a characterization is not possible. This problem is deeply connected with a recent conjecture relating the roots of the Steiner polynomial of a convex body and its inradius, namely, that given a convex body K with inradius 1, then -1 is an $(n-1)$-fold root of $f_{K;B_n}(z)$ if and only if K is a sausage with respect to B_n. This relation comes from the property, proved in this chapter, that linearity of the volume for K, E is equivalent to the fact that -1 is an $(n-1)$-fold root of the Steiner polynomial $f_{K;E}(z)$. Although this conjecture is known to be true in the plane for any arbitrary body E, as a consequence of the equality case in the well-known Bonnesen-Blaschke inequality, it will turn out to be false for higher dimensions in the most general setting: a counterexample for the case of an arbitrary gauge body is explicitly given.

In the same line, a counterexample to a conjecture by Matheron of 1978 on inner parallel bodies and the so-called *alternating Steiner polynomial* is also provided.

Afterwards we deal with the corresponding refinement of Minkowski's first inequality, when working with additional projections/sections assumptions and we will find out that sausages are again the bodies which allow to characterize the equality case. In particular, we obtain linear improvements of both the (classical) isoperimetric inequality and Urysohn's inequality and whose equality cases are given by (the family of) sausages with respect to a ball. The chapter ends with a characterization of the *linearity of the determinant* of positive definite symmetric matrices via 'sausages' of matrices, i.e., the sum of a matrix of rank (at most) 1 and another matrix; it may be seen as the counterpart to the linear behavior of the volume function, with the difference/advantage that no further assumptions (on 'projections/sections') are needed.

In the *last chapter*, we investigate the roots of a family of geometric polynomials of convex bodies associated to a given measure μ on the non-negative real line $\mathbb{R}_{\geq 0}$, which arise from a natural generalization of the Wills functional. We study its structure, showing that the set of roots in the upper half-plane is a closed convex cone, containing the non-positive real axis $\mathbb{R}_{\leq 0}$, and strictly increasing in the dimension, for any measure μ. Moreover, it is proved, on the one hand, that the 'smallest' cone of roots of these μ-polynomials is the one given by the Steiner polynomial, which provides, for example, additional information about the roots of μ-polynomials when the dimension is large enough. This will also imply geometric necessary conditions for a sequence $\{m_i : i = 0, 1, \dots\}$ to be the moments of a certain measure on $\mathbb{R}_{\geq 0}$, a question regarding the so-called (Stieltjes) moment problem. On the other hand, we also determine the μ-type polynomials which provide the 'biggest' cone of roots when working with log-concave measures.

In the second section of this chapter, we study properties of μ-polynomials associated to the unit p-balls, which may be regarded as the generalization for these bodies of the classical Wills polynomial. First, we show that the corresponding functional can be bounded just by the last but one relative quermassintegral. Then we also relate the roots of the Steiner and the Wills polynomials of unit p-balls, by giving a general asymptotic relation between the roots of Steiner polynomials and the above-mentioned polynomials, which provides another argument why the Wills functional may arise in a natural way. These properties will be obtained as consequences of more general results for some 'weighted'-Steiner polynomials, which will be referred throughout the dissertation as **m**-polynomials. Finally, we particularize the measure (and the gauge body) to investigate the roots of the classical Wills polynomial of a convex body. In particular, we give the precise description of the cones of roots for dimensions $n = 2, 3$ and we discuss some questions relative to the stability of the polynomial. Moreover, we study the size of its roots, bounding them in terms of functionals like the in- and circumradius, or the successive minima, of the set.

The original results which are contained in this dissertation can be found in the papers [29, 30, 31, 50, 62, 63].

Chapter 1

Preliminaries

In this first chapter we make a brief survey of the main definitions, properties and results of convex bodies and polynomials which will be needed throughout this dissertation.

1.1 Convex bodies and their properties

We will use the following standard notation. We write $\mathbb{R}^n$ to denote the n-dimensional Euclidean space, endowed with the standard inner product $\langle\cdot,\cdot\rangle$ and the Euclidean norm $|\cdot|$. We denote by e_i the i-th canonical unit vector in $\mathbb{R}^n$.

The closure of a set $M \subset \mathbb{R}^n$ is denoted by $\mathrm{cl}\, M$, its boundary by $\mathrm{bd}\, M$ and its interior by $\mathrm{int}\, M$. The *dimension* of a set $M \subset \mathbb{R}^n$, i.e., the dimension of the smallest affine subspace containing M (its *affine hull*, $\mathrm{aff}\, M$) is denoted by $\dim M$. Regarding the dimension of a convex set M, we write $\mathrm{relint}\, M$ to denote the relative interior of M, i.e., the interior of the set M relative to its affine hull.

Given $x, y \in \mathbb{R}^n$, $[x, y]$ will denote the segment determined by x and y, namely,

$$[x, y] = \big\{(1-\lambda)x + \lambda y : \, 0 \leq \lambda \leq 1\big\}.$$

The following definitions and properties are well known and can be found in any book on Convexity, for instance [9, 15, 21, 52, 55, 58]. We would like to mention also the work [3].

Definition 1.1. *A (non-empty) set $M \subset \mathbb{R}^n$ is said to be* convex *if, whenever two points $x, y \in M$, then the segment $[x, y]$ is contained in M, i.e., the convex combination $(1-\lambda)x + \lambda y \in M$, for $0 \leq \lambda \leq 1$.*

Definition 1.2. *A* convex body $K \subset \mathbb{R}^n$ *is a non-empty compact convex set, not necessarily with interior points.*

From now on $\mathcal{K}^n$ will denote the set of all convex bodies in $\mathbb{R}^n$. The subset of $\mathcal{K}^n$ consisting of all convex bodies with non-empty interior is denoted by $\mathcal{K}^n_0$.

The *Minkowski sum* of two convex bodies $K, L \in \mathcal{K}^n$ is nothing else but their (vectorial) addition, i.e.,

$$K + L = \{x + y : x \in K \text{ and } y \in L\},$$

which is clearly a convex body (see Figure 1.1), and we write $\lambda K = \{\lambda x : x \in K\}$, for $\lambda \in \mathbb{R}$. Two convex bodies $K, L \in \mathcal{K}^n$ are called *homothetic* if $K = \lambda L + t$ with $t \in \mathbb{R}^n$ and $\lambda \geq 0$.

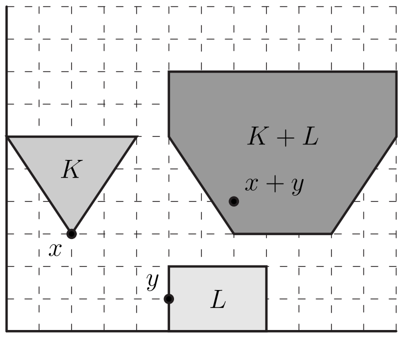

Figure 1.1: The Minkowski (vectorial) addition.

The intersection of all convex sets containing $M \subset \mathbb{R}^n$ is the *convex hull* of M, and it will be denoted by $\operatorname{conv} M$; thus $\operatorname{conv} M$ is the smallest convex set containing M. Analogously, the *linear hull* of M, $\operatorname{lin} M$, is defined, i.e., it is the intersection of all linear subspaces in $\mathbb{R}^n$ containing M. The convex hull of a compact set is always a convex body. In particular, the convex hull of a finite number of points is so and the family of all of them defines a very important class of convex bodies:

Definition 1.3. *A* polytope *is the convex hull of finitely many points in* $\mathbb{R}^n$ *(its* vertices*).*

A particular subfamily of polytopes that will be used along this dissertation are the *simplices*: an *n-simplex* is the convex hull of $n + 1$ affinely independent points in $\mathbb{R}^n$ (see Figure 1.2).

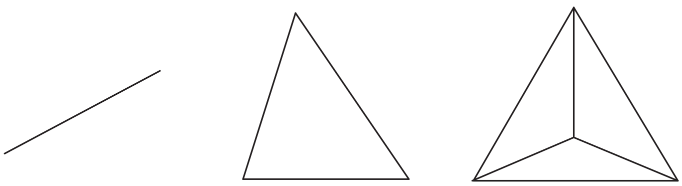

Figure 1.2: 1-, 2- and 3-simplices.

Furthermore, the set of all k-dimensional (linear) subspaces of $\mathbb{R}^n$ will be denoted by $\mathcal{L}^n_k$, and for $H \in \mathcal{L}^n_k$, with $H^\perp \in \mathcal{L}^n_{n-k}$ we represent the orthogonal complement of H. For $H \in \mathcal{L}^n_k$, $K \in \mathcal{K}^n$, the orthogonal projection of K onto H is denoted by $K|H$, which is a convex body as well.

In spite of the fact that many of the following properties and definitions are valid for closed convex sets, in order to simplify the exposition we will restrict them to compact ones, since we will always work under the hypothesis of compactness. An important notion is the following one:

Definition 1.4. *A hyperplane H is called a* supporting hyperplane *of $K \in \mathcal{K}^n$ if $H \cap K \neq \emptyset$ and K is contained in one of the two halfspaces determined by H, which is called its* supporting halfspace.

Supporting hyperplanes can be used to characterize convexity, because if $K \subset \mathbb{R}^n$ is a compact set with non-empty interior, then K is convex if and only if for every $x \in \operatorname{bd} K$ there exists a supporting hyperplane to K. As a consequence, we get that any convex body is the intersection of its supporting halfspaces.

There is no doubt that convex functions play an important role in the theory of convex bodies.

Definition 1.5. *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is* convex *if for any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$,*

$$f\big((1-\lambda)x + \lambda y\big) \leq (1-\lambda)f(x) + \lambda f(y).$$

Moreover, if $f\big((1-\lambda)x + \lambda y\big) < (1-\lambda)f(x) + \lambda f(y)$ for all $x, y \in \mathbb{R}^n$, $x \neq y$ and $0 < \lambda < 1$, then f is said to be strictly convex. *A function f is* concave *if $-f$ is convex, or equivalently, if for any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$, $f\big((1-\lambda)x + \lambda y\big) \geq (1-\lambda)f(x) + \lambda f(y)$, and it is* strictly concave *if $-f$ is strictly convex. Finally, f is* affine *if it is convex and concave.*

Related to the notion of concavity, we have the following definition, which will be used later on.

Definition 1.6. *A non-negative function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is said to be* log-concave *if its logarithm, $\log f$, is concave, i.e., if for all $x, y \in \operatorname{dom} f$ and all $\lambda \in [0,1]$, f satisfies*

$$f\big((1-\lambda)x + \lambda y\big) \geq f(x)^{1-\lambda} f(y)^\lambda.$$

The following properties of convex functions will be needed later. For references and further study we refer for instance to [48, 52].

Proposition 1.1.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex (concave) function. Then,*

i) *f is continuous in $\operatorname{int} \operatorname{dom} f$ and*

ii) *if $n = 1$, and f is twice-differentiable, then f is convex (concave) if and only if $f''(x) \geq 0$ ($f''(x) \leq 0$) for all $x \in \operatorname{dom} f$. Moreover, if $f''(x) > 0$ ($f''(x) < 0$) for all $x \in \operatorname{dom} f$ then f is strictly convex (strictly concave).*

We include the following remark on convex (concave) functions which turns out to be fundamental in some future proofs and is an easy consequence of the definition of concavity.

Remark 1.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a convex (concave) function such that, for some $\lambda_0 \in (0, 1)$, $f\big((1-\lambda_0)a+\lambda_0 b\big)= (1-\lambda_0)f(a)+\lambda_0 f(b)$. Then f is an affine function on the whole interval $[a, b]$.*

The following well-known result about sequences of functions will be needed later on.

Theorem 1.1.1. *Let $f_n : [a, b] \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions of class $\mathcal{C}^1$ such that there exists $x_0 \in [a, b]$ with $\big\{f_n(x_0)\big\}_n$ convergent, and such that $f'_n \to g$ converges uniformly. Then $f(x) = \lim_{n\to\infty} f_n(x)$ converges uniformly and $f'(x) = g(x) = \lim_{n\to\infty} f'_n(x)$.*

The space of convex bodies $\mathcal{K}^n$ is endowed with the *Hausdorff metric*, namely

$$\delta(K, L) = \min\big\{\lambda \geq 0 : K \subset L + \lambda B_n,\ L \subset K + \lambda B_n\big\} \quad \text{for} \ \ K, L \in \mathcal{K}^n,$$

where B_n denotes the n-dimensional Euclidean unit ball, which allows to consider continuity of functionals defined on $\mathcal{K}^n$ and approximation of convex bodies. We finish this section by formulating the famous *Blaschke selection theorem*, which provides a very useful tool in proving the existence of convex bodies with specific properties.

Theorem 1.1.2 (Blaschke's selection theorem). *Every bounded sequence of convex bodies in $\mathbb{R}^n$ has a convergent subsequence (in the Hausdorff metric) to a convex body.*

1.2 The Steiner formula and the Wills functional

The so-called *(relative) Steiner formula* of a convex body $K \in \mathcal{K}^n$, with respect to a *gauge body* $E \in \mathcal{K}^n$, is nothing else but a polynomial of degree at most n, which expresses the volume of the Minkowski sum of K and an homothetic copy of E with factor λ (the variable of the polynomial). In order to introduce the Steiner polynomial and the general setting involving the so-called mixed volumes, we need to define the volume.

Definition 1.7. *Given a convex body $K \in \mathcal{K}^n$, the* volume *of K is defined as its Lebesgue measure and will be denoted by $\mathrm{vol}(K)$ (or $\mathrm{vol}_n(K)$ if the distinction of the dimension is needed).*

Therefore, $\mathrm{vol}(\cdot)$ satisfies the known properties of the Lebesgue measure, namely:

i) If $\dim K = n$ then $\mathrm{vol}(K) > 0$. If $\dim K \leq n - 1$ then $\mathrm{vol}(K) = 0$.

ii) $\mathrm{vol}(\lambda K) = \lambda^n \mathrm{vol}(K)$ for $\lambda \geq 0$.

iii) The volume $\mathrm{vol} : \mathcal{K}^n \longrightarrow \mathbb{R}_{\geq 0}$ is a continuous function on the space of convex bodies.

iv) If $L \subset K$ then $\mathrm{vol}(L) \leq \mathrm{vol}(K)$ and equality holds, for $\dim L = n$, if and only if $L = K$.

Another relevant property of the volume is the following well-known result:

Theorem 1.2.1 (Fubini). *Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_k^n$. Then,*

$$\mathrm{vol}(K) = \int_{H^\perp} \mathrm{vol}_k\big(K \cap (t+H)\big)\,\mathrm{d}t.$$

Combining the notions of volume and Minkowski sum, the concept of mixed volume appears. For a deep study of mixed volumes we refer mainly to Section 5.1 in [52].

Theorem 1.2.2. *Let $K_1, \dots, K_m \in \mathcal{K}^n$ and $\lambda_i \geq 0$ for $i = 1, \dots, m$. Then*

$$\mathrm{vol}\left(\sum_{i=1}^m \lambda_i K_i\right) = \sum_{i_1=1}^m \cdots \sum_{i_n=1}^m \lambda_{i_1} \cdots \lambda_{i_n} \mathrm{V}(K_{i_1}, \dots, K_{i_n}). \tag{1.1}$$

The coefficients $\mathrm{V}(K_{i_1}, \dots, K_{i_n})$ are symmetric in the indices for any permutation, and they are called the mixed volumes *of $K_1, \dots, K_m$.*

Some useful properties of the mixed volumes are listed in the following proposition; they will be needed throughout this work.

Proposition 1.2.1. *Let $K, L, K_1, \dots, K_n \in \mathcal{K}^n$. The following properties hold:*

i) $\mathrm{V}(K, \dots, K) = \mathrm{vol}(K)$.

ii) $\mathrm{V}(rK + sL, K_2, \dots, K_n) = r\mathrm{V}(K, K_2, \dots, K_n) + s\mathrm{V}(L, K_2, \dots, K_n)$ *for every $r, s \geq 0$.*

iii) *Mixed volumes are continuous functions on $(\mathcal{K}^n)^n$, and translation and rigid motion invariant.*

iv) *If $K \subseteq L$ then $\mathrm{V}(K, K_2, \dots, K_n) \leq \mathrm{V}(L, K_2, \dots, K_n)$.*

v) $\mathrm{V}(K_1, \dots, K_n) \geq 0$. *Moreover, $\mathrm{V}(K_1, \dots, K_n) > 0$ if and only if $\dim(K_{i_1} + \dots + K_{i_k}) \geq k$ for each choice of indices $1 \leq i_1 < \dots < i_k \leq n$ and all $k \in \{1, \dots, n\}$.*

Another useful and important property of the mixed volumes, but now referring to sets lying in subspaces, is collected in the following theorem (see page 300, Theorem 5.3.1 and identity (5.68) in [52]). We denote by $\vartheta^{(k)}$ the mixed volume computed in a k-dimensional affine subspace.

Theorem 1.2.3. *Let $H \in \mathcal{L}_k^n$, for $k \in \{1, \dots, n-1\}$, and let $K_1, \dots, K_{n-k}, L_1, \dots, L_k \in \mathcal{K}^n$ with $L_i \subset H$ for all $i = 1, \dots, k$. Then*

$$\binom{n}{k}\mathrm{V}(K_1, \dots, K_{n-k}, L_1, \dots, L_k) = \vartheta^{(k)}(L_1, \dots, L_k)\,\vartheta^{(n-k)}\big(K_1|H^\perp, \dots, K_{n-k}|H^\perp\big).$$

In the particular case when $k = 1$, i.e., if L is a line segment, we get

$$n\mathrm{V}(K_1, \dots, K_{n-1}, L) = \mathrm{vol}_1(L)\,\vartheta^{(n-1)}\big(K_1|L^\perp, \dots, K_{n-1}|L^\perp\big). \tag{1.2}$$

The polynomial expansion (1.1) can be written in a more concise form. Introducing, for the sake of brevity, the notation

$$\big(K_1[r_1],\dots,K_m[r_m]\big)\equiv\Big(K_1,\overset{(r_1)}{\dots},K_1,\dots,K_m,\overset{(r_m)}{\dots},K_m\Big)$$

and the multinomial coefficient

$$\binom{n}{r_1\dots r_m}=\begin{cases}\dfrac{n!}{r_1!\cdots r_m!} & \text{if } \sum_{j=1}^m r_j=n, \text{ and } r_j\in\{0,\dots,n\}\\ 0 & \text{otherwise,}\end{cases}$$

using the linearity in each variable of mixed volumes, it is easy to check that

$$\mathrm{vol}\left(\sum_{i=1}^m\lambda_iK_i\right)=\sum_{r_1,\dots,r_m=0}^n\binom{n}{r_1\dots r_m}\lambda_1^{r_1}\cdots\lambda_m^{r_m}\mathrm{V}\big(K_1[r_1],\dots,K_m[r_m]\big).$$

Similar polynomial expansions are obtained for the mixed volume if some of its arguments are fixed. Let $p\in\{1,\dots,n\}$ and let $C_{p+1},\dots,C_n\in\mathcal{K}^n$ be given. Then

$$\begin{aligned}\mathrm{V}&\left(\sum_{i=1}^m\lambda_iK_i\,[p],C_{p+1},\dots,C_n\right)\\&=\sum_{r_1,\dots,r_m=0}^p\binom{p}{r_1\dots r_m}\lambda_1^{r_1}\cdots\lambda_m^{r_m}\mathrm{V}\big(K_1[r_1],\dots,K_m[r_m],C_{p+1},\dots,C_n\big).\end{aligned}\tag{1.3}$$

In the particular case of two convex bodies $K,E\in\mathcal{K}^n$, the mixed volumes $\mathrm{V}\big(K[n-i],E[i]\big)$, for $i=0,\dots,n$, are called the *relative quermassintegrals* of K (with respect to E) and they are denoted by $\mathrm{W}_i(K;E)$. In particular, $\mathrm{W}_0(K;E)=\mathrm{vol}(K)$, $\mathrm{W}_n(K;E)=\mathrm{vol}(E)$ and moreover,

$$\mathrm{W}_i(K;E)=\mathrm{W}_{n-i}(E;K).$$

When $E=B_n$ the *(classical)* i-th quermassintegral $\mathrm{W}_i(K)=\mathrm{W}_i(K;B_n)$ is just called i-th *quermassintegral* of K. In particular $\mathrm{W}_0(K)=\mathrm{vol}(K)$, $\mathrm{W}_n(K)=\mathrm{vol}(B_n)$, $n\mathrm{W}_1(K)=\mathrm{S}(K)$ is the usual *surface area* of K and $\big(2/\mathrm{vol}(B_n)\big)\mathrm{W}_{n-1}(K)=\mathrm{b}(K)$ is the *mean width* of K (see e.g. (1.30) in [52]). If $n=2$, then $\mathrm{vol}(K)=\mathrm{A}(K)$ is called the *area* of K and $2\mathrm{W}_1(K)=\mathrm{p}(K)$ is its *perimeter*.

If we have to distinguish the dimension in which the quermassintegrals are computed, we will write $\mathrm{W}_i^{(k)}$ to denote the i-th quermassintegral in $\mathbb{R}^k$. Then, the following connection between the quermassintegrals of a convex body $K\in\mathcal{K}^n$ with $\dim K=k$ $(k\le n)$, computed, respectively, in $\mathbb{R}^n$ and $\mathbb{R}^k$, holds (see e.g. Property 3.1 in [49]):

$$\mathrm{W}_{n-k+i}(K)=\frac{\kappa_{n-k+i}}{\kappa_i}\frac{\binom{k}{i}}{\binom{n}{k-i}}\mathrm{W}_i^{(k)}(K),\qquad i=0,\dots,k,\tag{1.4}$$

where we write $\kappa_n=\mathrm{vol}(B_n)$ (see (1.15) for additional properties). The remaining quermassintegrals in $\mathbb{R}^n$, namely, $\mathrm{W}_j(K)$ for $j=0,\dots,n-k-1$, vanish.

Taking into account the following definition, the so-called *relative Steiner formula* or *Minkowski-Steiner formula* is obtained (cf. Theorem 1.2.2).

Definition 1.8. *For $K \in \mathcal{K}^n$, the* outer parallel body (relative to E) *of K at distance $\lambda \geq 0$ is the Minkowski sum $K + \lambda E$.*

Theorem 1.2.4 (The (relative) Steiner formula. Steiner, [54]). *Let $K, E \in \mathcal{K}^n$. The volume of the outer parallel body of K with respect to E at distance $\lambda \geq 0$ is expressed as*

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)\lambda^i. \tag{1.5}$$

We notice that if $E \in \mathcal{K}_0^n$, then the polynomial in the right-hand side of (1.5), the so-called *relative Steiner polynomial*, has degree n, i.e., the dimension of the space. Moreover, the formal polynomial expression in the complex variable $z \in \mathbb{C}$

$$f_{K;E}(z) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)z^i, \tag{1.6}$$

is (also) known as the (relative) Steiner polynomial of K with respect to E. We notice that for $z \in \mathbb{R}$, $z \geq 0$, it provides the volume of $K + zE$ (cf. (1.5)).

Taking into account that quermassintegrals are particular cases of mixed volumes, the following *Steiner formula* for the relative quermassintegrals can be obtained.

Theorem 1.2.5 (Steiner formula for relative quermassintegrals). *Let $K, E \in \mathcal{K}^n$ and $\lambda \geq 0$. The relative i-th quermassintegral, $i = 0, \dots, n$, of the outer parallel body of K (relative to E), $K + \lambda E$, can be expressed as a polynomial in the parameter λ,*

$$\mathrm{W}_i(K + \lambda E; E) = \sum_{k=0}^{n-i} \binom{n-i}{k} \mathrm{W}_{i+k}(K; E)\lambda^k. \tag{1.7}$$

In the particular case $E = B_n$, (1.5) becomes the *classical Steiner formula* of K:

$$\mathrm{vol}(K + \lambda B_n) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K)\lambda^i. \tag{1.8}$$

In [42], McMullen considered the normalized quermassintegrals

$$\mathrm{V}_i(K) = \binom{n}{i} \frac{\mathrm{W}_{n-i}(K)}{\kappa_{n-i}}, \tag{1.9}$$

and proposed to call these measures the *intrinsic volumes* of K, since, if K is k-dimensional, then $\mathrm{V}_k(K)$ is the usual k-dimensional volume of K. The intrinsic volumes depend only on the convex body K but not on the dimension of the embedding space (see e.g. Section 6.4 in [21]). Thus the Steiner formula (1.8) can be represented via (1.9) as

$$\mathrm{vol}(K + \lambda B_n) = \sum_{i=0}^{n} \kappa_i \mathrm{V}_{n-i}(K)\lambda^i.$$

In 1973, Wills [59] introduced and studied the functional

$$W(K)=\sum_{i=0}^{n}\mathrm{V}_i(K)=\sum_{i=0}^{n}\binom{n}{i}\frac{\mathrm{W}_i(K)}{\kappa_i} \tag{1.10}$$

because of *its possible relation* with the so-called *lattice-point enumerator*

$$G(K)=\#(K\cap\mathbb{Z}^n),$$

i.e., the number of points with integer coordinates contained in K, and conjectured that $W(K)$ is bounded by above by $G(K)$. Although Hadwiger [23] showed that Wills' conjecture was wrong (see also [6]), the *Wills functional* turned out to have several interesting applications, e.g., in Discrete Geometry, where there exist nice relations of this functional with the so-called *successive minima*, which we introduce next, of a convex body [61], or in deriving exponential moment inequalities for Gaussian random processes [56]. Many other nice properties of this functional, as well as relations with other measures, have been studied in the last years, see e.g. [22, 23, 43, 59, 60, 61]. More recently, the Wills functional has been also considered from a more general point of view or in a probabilistic context (see [32] and [56, 57], respectively).

In [22] Hadwiger showed, among others, the following integral representations of $W(K)$:

$$W(K)=\int_{\mathbb{R}^n}e^{-\pi d(x,K)^2}\mathrm{d}x,\quad W(K)=2\pi\int_0^\infty \mathrm{vol}(K+tB_n)t\,e^{-\pi t^2}\mathrm{d}t, \tag{1.11}$$

where $d(x,K)$ denotes the Euclidean distance between $x\in\mathbb{R}^n$ and K.

1.3 Some relevant functionals and convex bodies

We include here a couple of additional definitions that we will use throughout this dissertation.

Definition 1.9. *Let $K\in\mathcal{K}^n$. The* relative inradius $\mathrm{r}(K;E)$ *of K with respect to $E\in\mathcal{K}^n$ and the* relative circumradius $\mathrm{R}(K;E)$ *of K with respect to $E\in\mathcal{K}^n_0$ are defined, respectively, by*

$$\mathrm{r}(K;E)=\max\bigl\{\lambda\ge 0:\ x+\lambda E\subset K \textit{ for some } x\in\mathbb{R}^n\bigr\},$$
$$\mathrm{R}(K;E)=\min\bigl\{\lambda\ge 0:\ K\subset x+\lambda E \textit{ for some } x\in\mathbb{R}^n\bigr\}.$$

Moreover, such points x in the above expressions are called, respectively, the (relative) incenter *and* circumcenter *of K. Moreover, the* diameter *of K is defined as* $\mathrm{D}(K)=\max\bigl\{|x-y|:x,y\in K\bigr\}$.

We notice that the relation

$$\mathrm{r}(K;E)\mathrm{R}(E;K)=1.$$

always holds. In the particular case when $E=B_n$ the classical inradius $\mathrm{r}(K)=\mathrm{r}(K;B_n)$ and circumradius $\mathrm{R}(K)=\mathrm{R}(K;B_n)$ are obtained (see Figure 1.3).

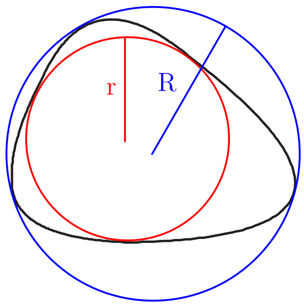

Figure 1.3: The classical inradius and circumradius.

Since, up to translations, $\mathrm{r}(K)B_n \subset K$ and $K \subset \mathrm{R}(K)B_n$ the following inequalities are a direct consequence of the monotonicity of the mixed volumes:

$$\mathrm{r}(K)\mathrm{W}_{i+1}(K) \leq \mathrm{W}_i(K) \leq \mathrm{R}(K)\mathrm{W}_{i+1}(K), \tag{1.12}$$

for all $i = 0, \ldots, n-1$.

As already introduced in the previous page, we denote by $\mathbb{Z}^n$ the integer lattice, i.e., the set of all points with integer coordinates in $\mathbb{R}^n$. As a general reference for lattices and successive minima we refer to [21].

Definition 1.10. *Let $K \in \mathcal{K}^n$ be a 0-symmetric convex body, i.e., satisfying that $K = -K$. The i-th* successive minimum *$\lambda_i(K)$ of K, $i = 1, \ldots, n$, is defined as*

$$\lambda_i(K) = \min\bigl\{\lambda > 0 : \dim(\lambda K \cap \mathbb{Z}^n) \geq i\bigr\}.$$

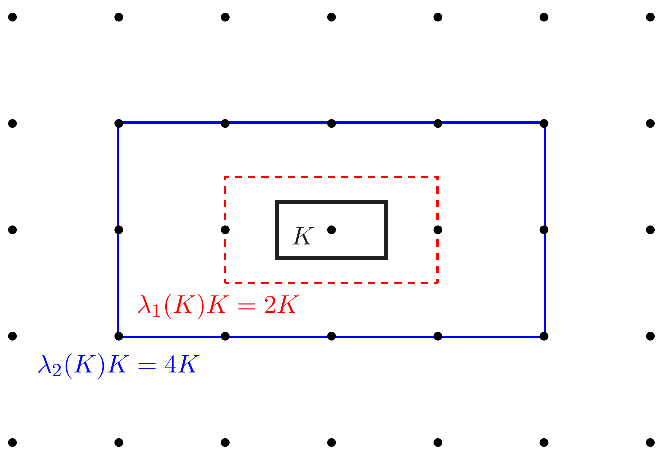

Figure 1.4: The successive minima of a rectangle.

Clearly the successive minima form an increasing sequence, i.e., $0 < \lambda_1(K) \leq \cdots \leq \lambda_n(K)$, and they are homogeneous of degree -1, this is, $\lambda_i(\alpha K) = (1/\alpha)\lambda_i(K)$.

1.3.1 Special convex bodies

A particularly interesting family of convex bodies is the one of the p-balls: for $1 \le p \le \infty$, we write B_n^p to represent the unit p-ball associated to the p-norm $|\cdot|_p$ (see Figure 1.5), namely,

$$B_n^p = \left\{ x = (x_1, \dots, x_n)^t \in \mathbb{R}^n : |x|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \le 1 \right\}, \tag{1.13}$$

where, as usual $|x|_\infty$ must be understood as $|x|_\infty = \max_{1 \le i \le n} |x_i|$.

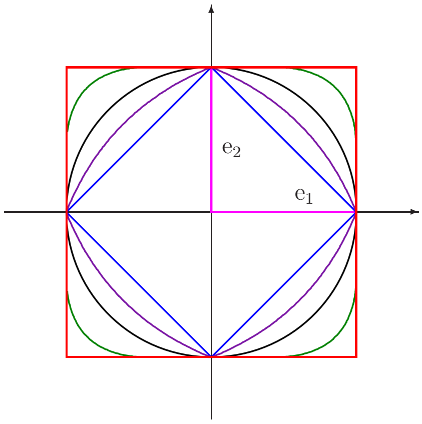

Figure 1.5: The p-balls.

In the particular case $p = 2$, we write for short $B_n^2 = B_n$ and $\mathbb{S}^{n-1}$ for the $(n-1)$-dimensional unit sphere, i.e., its boundary. We also denote by C_n the n-dimensional cube of edge-length 1 centered at the origin, i.e., $C_n = (1/2)\, B_n^\infty$.

We write $\kappa_n^p = \mathrm{vol}(B_n^p)$, which takes the value

$$\kappa_n^p = \frac{\left(2\Gamma\left(\frac{1}{p} + 1\right) \right)^n}{\Gamma\left(\frac{n}{p} + 1\right)} \tag{1.14}$$

(see e.g. page 11 of [47]), where Γ denotes the *gamma function*, namely,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t$$

for any $x > 0$. Thus, the volume of the unit Euclidean ball is given by

$$\kappa_n = \mathrm{vol}(B_n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}. \tag{1.15}$$

By (1.15) and the fact that $\Gamma(t+1) = t\Gamma(t)$ for $t > 0$ (for some properties of the gamma function see e.g. Section 5.3 in [58]), it is obtained that

$$\frac{\kappa_n}{\kappa_{n-2}} = \frac{2\pi}{n}. \tag{1.16}$$

Proposition 1.3.1. *The gamma function satisfies*

$$\Gamma(x) = \sqrt{2\pi}\, x^{x-1/2} e^{-x+\eta(x)}, \tag{1.17}$$

for all $x > 0$*, where*

$$\eta(x) = \sum_{k=0}^{\infty} \left(\left(x + k + \frac{1}{2} \right) \log\left(1 + \frac{1}{x+k} \right) - 1 \right).$$

Since it can be checked that

$$\eta(x) = \frac{\theta(x)}{12x}, \qquad 0 < \theta(x) < 1,$$

formula (1.17) yields the asymptotic formula (see e.g. Theorem 5.3.12 in [58] and page 24 of [2])

$$\lim_{n\to\infty} \frac{\Gamma(x_n)}{\sqrt{2\pi} \left(\frac{x_n}{e}\right)^{x_n} \frac{1}{\sqrt{x_n}}} = 1, \tag{1.18}$$

provided that $(x_n)_{n\in\mathbb{N}} \to \infty$ if n goes to ∞. Both relations (1.17) and (1.18) are usually called *Stirling's formulae* and allow to estimate $\Gamma(x)$ for large values of $x > 0$.

Moreover, the well-known *beta function* (see e.g. page 215 of [58])

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t,$$

for $x, y > 0$, satisfies the relations

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2 \int_0^{\pi/2} (\sin s)^{2x-1} (\cos s)^{2y-1} \mathrm{d}s.$$

Another family of convex bodies which will be often used along this dissertation is that of the so-called p-tangential bodies. Tangential bodies can be defined in several equivalent ways; here we will use the following one. For further characterizations and properties of p-tangential bodies we refer to Section 2.2 in [52].

The *normal cone* $N(K, x)$ of $K \in \mathcal{K}^n$ at $x \in \operatorname{bd} K$ consists of all outer normal vectors of K at x together with the zero vector. Then the boundary point x is said to be an *r-singular point* of K if the dimension $\dim N(K, x) \geq n - r$.

Definition 1.11. *A convex body* $K \in \mathcal{K}^n$ *containing the convex body* $E \in \mathcal{K}^n$ *is called a* p-tangential body *of* E*,* $p \in \{0, \dots, n-1\}$*, if each support plane of* K *not supporting* E *contains only* $(p-1)$*-singular points of* K*.*

So a 0-tangential body of E is just the body E itself and each p-tangential body of E is also a q-tangential body for $p < q \leq n-1$. An $(n-1)$-tangential body will be briefly called *tangential body.*

A 1-tangential body is usually called *cap-body*, and it can be seen as the convex hull of E and countably many points such that the line segment joining any pair of those points intersects E (see Figure 1.6, left). The n-dimensional cube C_n is an example of tangential body.

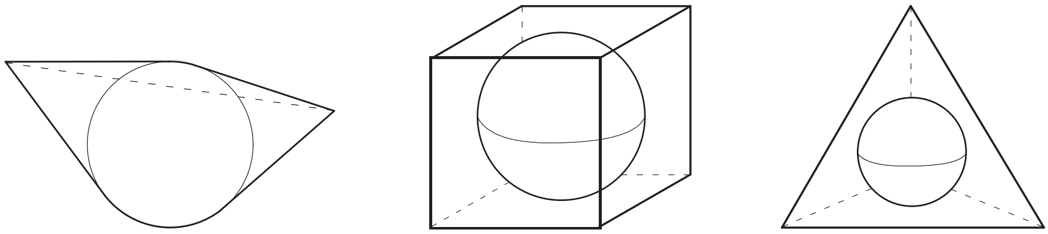

Figure 1.6: The cap bodies (left) and the regular cube and simplex are tangential bodies of B_n.

We state here the following result by Favard [16] that allows to characterize an n-dimensional p-tangential body in terms of the relative quermassintegrals, and which will be needed along this work (see also Theorem 7.6.17 in [52]).

Theorem 1.3.1 (Favard, [16]). *Let $K, E \in \mathcal{K}^n_0$, $E \subset K$, and let $p \in \{0, \dots, n-1\}$. Then*

$$\mathrm{W}_0(K; E) = \mathrm{W}_1(K; E) = \dots = \mathrm{W}_{n-p}(K; E)$$

if and only if K is a p-tangential body of E.

Remark 1.2. *We notice that, in the particular case when $E = \mathrm{r}(K)B_n$, a convex body $K \in \mathcal{K}^n$ containing the ball $\mathrm{r}(K)B_n$ is a tangential body, if and only if the equality $\mathrm{W}_0(K) = \mathrm{r}(K)\mathrm{W}_1(K)$ holds (cf. (1.12), $i = 0$).*

Another relevant family of convex bodies are the so-called *sausages* since, as we will show throughout this work, they will essentially characterize some linear Brunn-Minkowski type inequalities under certain projection assumptions on convex bodies.

Definition 1.12. *We say that a pair of convex bodies $K, E \in \mathcal{K}^n$ is a* sausage *if there exists $L \in \mathcal{K}^n$ with $\dim L \leq 1$ such that either $K = L + E$ or $E = L + K$. In particular, K is a sausage with respect to B_n if $K = L + B_n$.*

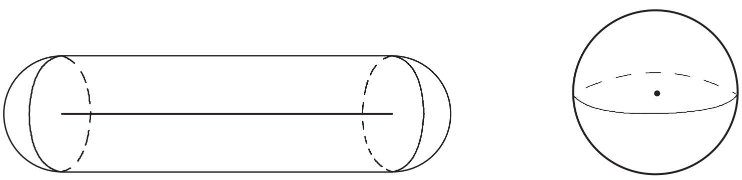

Figure 1.7: Sausages with respect to a ball.

Another important notion that will play a key role in this work is the *Schwarz symmetrization* of a convex body K with respect to a k-plane $H \in \mathcal{L}^n_k$ (see chapter IV of [36], page 58 of [18]), which is defined as follows:

Definition 1.13. *For any* $y \in K|H$*, let* $B_{n-k}(y, r_y) \subset y + H^{\perp}$ *be the* $(n-k)$*-dimensional Euclidean ball with center* y *and radius* r_y*, such that* $\mathrm{vol}_{n-k}\big(B_{n-k}(y, r_y)\big) = \mathrm{vol}_{n-k}\big(K \cap (y + H^{\perp})\big)$*; then*

$$\sigma_H(K) = \bigcup_{y \in K|H} B_{n-k}(y, r_y)$$

is the Schwarz symmetral of K with respect to H.

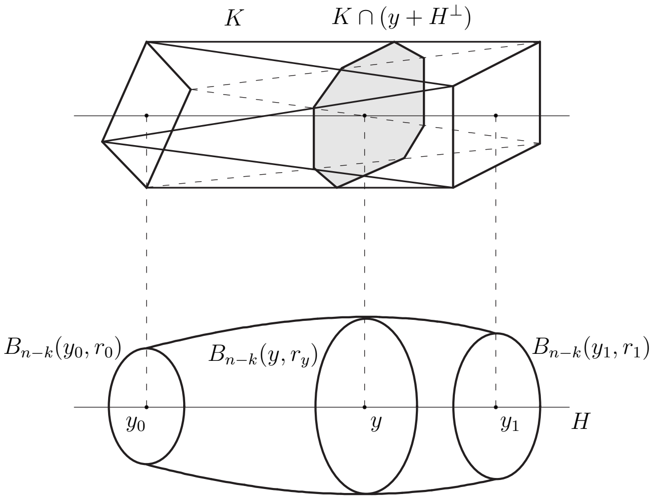

Figure 1.8: The Schwarz symmetrization.

Next lemma collects some properties of the Schwarz symmetrization that will be needed later.

Lemma 1.3.1. *Let* $K, E \in \mathcal{K}^n$ *and* $H \in \mathcal{L}^n_k$*. Then:*

i) $\mathrm{vol}(K) = \mathrm{vol}\big(\sigma_H(K)\big)$.

ii) $\sigma_H\big(\alpha K + \beta E\big) \supset \alpha\sigma_H(K) + \beta\sigma_H(E)$*, for* $\alpha, \beta \geq 0$.

iii) $K|H = \sigma_H(K)|H = \sigma_H(K) \cap H$.

1.4 Inequalities for mixed volumes and other related results

Mixed volumes satisfy many inequalities. Here we collect some of the most relevant ones, which will be needed throughout this work. One of them has been already stated: inequality (1.12).

We dare to say that the most important inequality relating mixed volumes is the *Aleksandrov-Fenchel inequality.* For a deep study of this inequality we refer to Sections 7.3, 7.6 in [52].

Theorem 1.4.1 (Aleksandrov-Fenchel inequality). *Let* $K_1, \dots, K_n \in \mathcal{K}^n$*. Then*

$$\mathrm{V}(K_1, K_2, K_3, \dots, K_n)^2 \geq \mathrm{V}(K_1, K_1, K_3, \dots, K_n)\mathrm{V}(K_2, K_2, K_3, \dots, K_n). \qquad (1.19)$$

Clearly, equality holds in (1.19) if K_1 and K_2 are homothetic. However, the complete classification of the equality case has not been settled yet. Only in several special cases the solution is known.

As particular cases of the most general Aleksandrov-Fenchel inequality (1.19) we get the so-called Aleksandrov-Fenchel inequalities for quermassintegrals, which turn out to be a very useful tool for some of the proofs along this dissertation: for all $i = 1, \dots, n-1$,

$$\mathrm{W}_i(K;E)^2 \geq \mathrm{W}_{i-1}(K;E)\mathrm{W}_{i+1}(K;E) \tag{1.20}$$

or, more generally, it can be also deduced that

$$\mathrm{W}_i(K;E)\mathrm{W}_j(K;E) \geq \mathrm{W}_{i-1}(K;E)\mathrm{W}_{j+1}(K;E), \quad 1 \leq i \leq j \leq n-1. \tag{1.21}$$

Theorem 4 in [53] shows that these inequalities allow to characterize quermassintegrals/Steiner polynomials:

Theorem 1.4.2 (Shephard, [53]). *If a sequence of real numbers $a_0, \dots, a_n \geq 0$ satisfies inequalities* (1.21)*, then there exist simplices $K, E \in \mathcal{K}^n$ such that $\mathrm{W}_i(K;E) = a_i$.*

Shephard gave an explicit construction of the two simplices in the case when all $\mathrm{W}_i > 0$ (see Figure 1.9), whereas the general case was obtained by a rather non-constructive topological argument. In Lemma 2.1 of [27] the authors reduced the number of involved inequalities, and extended the construction of the two convex bodies to the case $\mathrm{W}_i \geq 0$ (see Subsection 4.1.2 for a more detailed explanation).

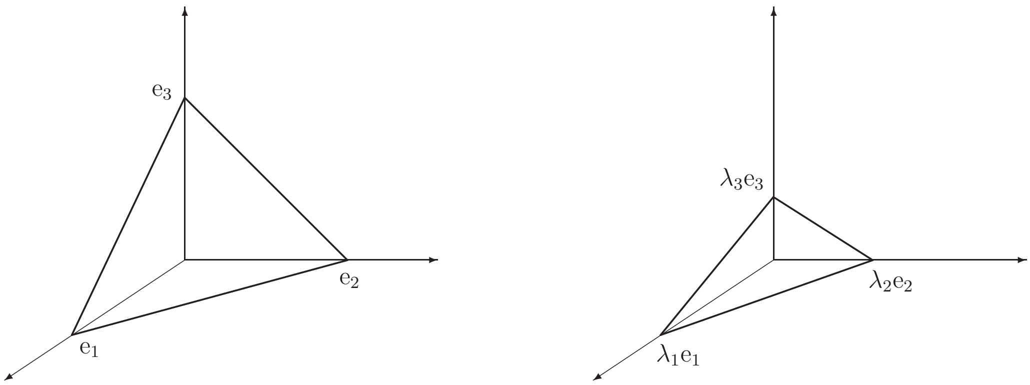

Figure 1.9: Construction of the simplices for given quermassintegrals: $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

Regarding the equality case in the above inequalities, we will need the following particular characterizations. For the first result, we refer to Theorem 7.6.2 in [52].

Theorem 1.4.3. *If $K, L \in \mathcal{K}^n$ are convex bodies for which equality holds in*

$$\mathrm{V}(K, L, B_n, \dots, B_n)^2 \geq \mathrm{V}(K, K, B_n, \dots, B_n)\mathrm{V}(L, L, B_n, \dots, B_n),$$

then K and L are homothetic.

Next result can be found in [52], Theorem 7.6.20.

Theorem 1.4.4. *If* $K \in \mathcal{K}^n$ *is* 0*-symmetric and* $i \in \{1, \dots, n-1\}$*, then the equality*

$$\mathrm{W}_i(K)^2 = \mathrm{W}_{i-1}(K)\mathrm{W}_{i+1}(K)$$

holds if and only if either $\dim K < n - i$ *or* K *is an* $(n-i-1)$*-tangential body of a ball.*

Relating the volume with the Minkowski addition of convex bodies, one is led to the famous Brunn-Minkowski inequality. Its statement is rather simple: it ensures the concavity of the n-th root of the volume functional, $\mathrm{vol}^{1/n} : \mathcal{K}^n \longrightarrow \mathbb{R}$:

Theorem 1.4.5 (Brunn-Minkowski's inequality). *For convex bodies* $K, L \in \mathcal{K}^n$ *and* $\lambda \in [0, 1]$*,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/n} \geq (1-\lambda)\mathrm{vol}(K)^{1/n} + \lambda \mathrm{vol}(L)^{1/n}, \tag{1.22}$$

i.e., the n*-th root of the volume is a concave function. Equality for some* $\lambda \in (0, 1)$ *holds if and only if* K *and* L *either lie in parallel hyperplanes or are homothetic.*

This theorem can be found in any of the already mentioned books of classical Convexity. Although the inequality is also true for the more general case of measurable sets, since our approach relies on convexity, we will make use of the above version. There is an equivalent *multiplicative version* of the Brunn-Minkowski inequality (see e.g. Theorem 8.15 in [21]):

Theorem 1.4.6 (Brunn-Minkowski's inequality, multiplicative version). *Let* $K, L \in \mathcal{K}^n$ *be convex bodies. Then*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda} \quad \textit{for} \ \ 0 \leq \lambda \leq 1.$$

Brunn-Minkowski's inequality is one of the most powerful results in Convex Geometry and beyond: for instance, its equivalent analytic version (Prékopa-Leindler's inequality, Theorem 1.4.8) and the fact that the convexity/compactness assumption can be 'weakened' to consider just Lebesgue measurable sets (see [37]), have allowed it to move in much wider fields. It implies very important inequalities as the isoperimetric and Urysohn inequalities (see e.g. page 382 in[52]) or even the Aleksandrov-Fenchel inequality, and it has been the starting point for new developments like the so-called L_p-Brunn-Minkowski theory (see e.g. [38, 39]), a Brunn-Minkowski result for integer lattices (see [19]), or a reverse Brunn-Minkowski inequality (see e.g. [44]), among many others. It would not be possible to collect here all known references regarding versions, applications and/or generalizations on Brunn-Minkowski's inequality. So, for extensive and beautiful surveys on them we refer to [5, 17].

Brunn-Minkowski's inequality has a more general version for quermassintegrals: if $K, L, E \in \mathcal{K}^n$ and $0 \leq \lambda \leq 1$, then, for all $i = 0, \dots, n-2$,

$$\mathrm{W}_i\big((1-\lambda)K + \lambda L; E\big)^{1/(n-i)} \geq (1-\lambda)\mathrm{W}_i(K; E)^{1/(n-i)} + \lambda \mathrm{W}_i(L; E)^{1/(n-i)}, \tag{1.23}$$

whereas

$$\mathrm{W}_{n-1}\big((1-\lambda)K+\lambda L;E\big)=(1-\lambda)\mathrm{W}_{n-1}(K;E)+\lambda\mathrm{W}_{n-1}(L;E).$$

In fact, there exist the most general version of Brunn-Minkowski's inequality for mixed volumes (Theorem 7.4.5 in [52]): for $m\in\{1,\dots,n\}$ and $K,L,K_{m+1},\dots,K_n\in\mathcal{K}^n$ given, the function

$$f(\lambda)=\mathrm{V}\big((1-\lambda)K+\lambda L[m],K_{m+1},\dots,K_n\big)^{1/m} \quad \text{is concave on } [0,1]. \tag{1.24}$$

Using Brunn-Minkowski's inequality and the fact that the volume of the convex combination $(1-t)K+tL$ is a polynomial in $t\in[0,1]$, another two important inequalities can be obtained, namely, the *first* and the *second Minkowski inequalities* (see Section 7.2 in [52], Theorem 7.2.1).

Theorem 1.4.7 (Minkowski's inequalities). *Let $K,L\in\mathcal{K}^n$. Then*

$$\mathrm{V}\big(K[n-1],L\big)^n\geq \mathrm{vol}(K)^{n-1}\mathrm{vol}(L). \qquad \text{(1st Minkowski's ineq.)} \tag{1.25}$$

For $K,L\in\mathcal{K}^n_0$, equality holds if and only if K and L are homothetic.

$$\mathrm{V}\big(K[n-1],L\big)^2\geq \mathrm{vol}(K)\mathrm{vol}\big(K[n-2],L[2]\big). \qquad \text{(2nd Minkowski's ineq.)}$$

For $L\in\mathcal{K}^n_0$, equality holds if and only if either $\dim K<n-1$ *or K is homothetic to an $(n-2)$-tangential body of L.*

We observe that second Minkowski's inequality is a particular case of the Aleksandrov-Fenchel inequality (1.19). Moreover, in the special case when $L=B_n$ is the unit ball, first Minkowski's inequality reduces to the famous *isoperimetric inequality*,

$$\mathrm{S}(K)^n\geq n^n\kappa_n\mathrm{vol}(K)^{n-1}. \tag{1.26}$$

1.4.1 Some functional inequalities

As we have mentioned before, the integral version of the Brunn-Minkowski inequality is the so-called *Prékopa-Leindler inequality*, which can be found, e.g., in [21], Theorem 8.14.

Theorem 1.4.8 (Prékopa-Leindler's inequality). *Let $\lambda\in(0,1)$ and $f,g,h:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq 0}$ be non-negative Lebesgue measurable functions such that, for any $x,y\in\mathbb{R}^n$,*

$$h\big((1-\lambda)x+\lambda y\big)\geq f(x)^{1-\lambda}g(y)^{\lambda}.$$

Then

$$\int_{\mathbb{R}^n}h(x)\,\mathrm{d}x\geq\left(\int_{\mathbb{R}^n}f(x)\,\mathrm{d}x\right)^{1-\lambda}\left(\int_{\mathbb{R}^n}g(x)\,\mathrm{d}x\right)^{\lambda}. \tag{1.27}$$

The following inequality can be seen as the reverse inequality to Prékopa-Leindler's theorem. It can be found, e.g., in [21], Corollary 1.5.

Theorem 1.4.9 (Hölder's inequality). *For a measure space, let $f, g : \Omega \longrightarrow \mathbb{R}$ be non-negative and integrable functions and let $1 \leq p, q \leq \infty$ be such that $1/p + 1/q = 1$. Then*

$$\int_\Omega f(x)g(x)\,\mathrm{d}\mu(x) \leq \left(\int_\Omega f(x)^p \mathrm{d}\mu(x)\right)^{1/p} \left(\int_\Omega f(x)^q \mathrm{d}\mu(x)\right)^{1/q}. \tag{1.28}$$

A particularly interesting case arises when $p = 2$:

Theorem 1.4.10 (Cauchy-Schwarz inequality). *For a measure space, let $f, g : \Omega \longrightarrow \mathbb{R}$ be non-negative and integrable functions. Then*

$$\int_\Omega f(x)g(x)\,\mathrm{d}\mu(x) \leq \left(\int_\Omega f(x)^2 \mathrm{d}\mu(x)\right)^{1/2} \left(\int_\Omega f(x)^2 \mathrm{d}\mu(x)\right)^{1/2}. \tag{1.29}$$

Equality holds if and only if there exists $\alpha > 0$ such that $f(x) = \alpha g(x)$ almost everywhere on Ω.

The *arithmetic-geometric mean inequality* (see e.g. Corollary 1.2 in [21]) is closely related with the above inequalities. We collect it here for future references.

Theorem 1.4.11 (Arithmetic-geometric mean inequality). *Let $x_1, \ldots, x_n$ be non-negative real numbers and let $\lambda_1, \ldots, \lambda_n \geq 0$ be such that $\sum_{i=1}^n \lambda_i = 1$. Then*

$$x_1^{\lambda_1} \cdots x_n^{\lambda_n} \leq \lambda_1 x_1 + \cdots + \lambda_n x_n. \tag{1.30}$$

1.5 Some properties on real polynomials

In the last section of this introductory chapter, we briefly collect some known results about (the roots of) real polynomials that will be needed throughout this dissertation. Most of them can be found in the book [40].

In the following, and unless we explicitly say the opposite, we will assume that any formal polynomial $\sum_{i=0}^n a_i z^i$ in the complex variable $z \in \mathbb{C}$ is a *real polynomial*, i.e., the coefficients $a_i \in \mathbb{R}$ for all $i = 0, \ldots, n$, since it is the setting which will be used in this work.

The first result allows to assert, roughly speaking, that the roots of a polynomial are continuous functions of the coefficients (see e.g. Theorem (1,4) of [40]):

Theorem 1.5.1. *Let*

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n = a_n \prod_{j=1}^p (z - z_j)^{m_j}, \qquad a_n \neq 0,$$

$$F(z) = (a_0 + \varepsilon_0) + \cdots + (a_{n-1} + \varepsilon_{n-1}) z^{n-1} + a_n z^n$$

and let

$$0 < r_k < \min |z_k - z_j|, \qquad j = 1, \ldots, p, \quad j \neq k.$$

Then there exists $\varepsilon > 0$ such that, if $|\varepsilon_i| \leq \varepsilon$ for $i = 0, \ldots, n-1$, $F(z)$ has precisely m_k zeros in the circle centered at z_k and with radius r_k.

We will also need the following *stability* criterion (see Theorem 3 in [45] and Theorem 1 in [33]):

Theorem 1.5.2 (Nie et al., [45]; Katkova et al., [33]). *A polynomial $f(z) = \sum_{i=0}^{n} a_i\, z^i$, with $a_i > 0$ for $i = 0, \dots, n$, is* stable, *i.e., all its roots have negative real part, if*

$$a_{i-1}a_{i+2} \le \beta\, a_i a_{i+1}, \quad i = 1, \dots, n-2, \tag{1.31}$$

where $\beta \approx 0.4655$ is the only real solution of $z(z+1)^2 = 1$.

Another useful stability criterion is the *Routh-Hurwitz criterion* (see Corollary (40,2) of [40]):

Theorem 1.5.3 (Routh-Hurwitz criterion). *Let $F(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ and let*

$$\delta_k = \det \begin{pmatrix} a_1 & a_3 & a_5 & \cdots & a_{2k-1} \\ 1 & a_2 & a_4 & \cdots & a_{2k-2} \\ 0 & a_1 & a_3 & \cdots & a_{2k-3} \\ 0 & 1 & a_2 & \cdots & a_{2k-4} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_k \end{pmatrix}$$

for $k = 1, \dots, n$, where $a_j = 0$ for $j > n$. If all the above determinants δ_k are positive then the polynomial $F(z)$ has only roots with negative real parts.

Regarding a 'more concrete' location of the roots of a real polynomial, we have the following result (see e.g. Exercise 2, page 137 of [40]):

Theorem 1.5.4. *The roots of a polynomial $\sum_{j=0}^{n} a_j z^j$ with coefficients $a_j > 0$ lie in the ring*

$$\min\left\{ \frac{a_j}{a_{j+1}} : j = 0, 1, \dots, n-1 \right\} \le |z| \le \max\left\{ \frac{a_j}{a_{j+1}} : j = 0, 1, \dots, n-1 \right\}.$$

The following well-known result can be found in [40], Theorem 6.1.

Theorem 1.5.5 (Gauss-Lucas theorem). *All the roots of the derivative of a non-constant polynomial $f(z)$ lie in the convex hull of the set of roots of $f(z)$.*

Next result can be found in [40], Theorem 16.1.

Theorem 1.5.6. *Let*

$$f(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k, \quad g(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k, \quad h(z) = \sum_{k=0}^{n} \binom{n}{k} a_k b_k z^k.$$

If all the roots of $f(z)$ lie in a set $K \subset \mathbb{C}$ which is either the closed interior or exterior of a circle or a closed half-plane, and if $\alpha_1, \dots \alpha_n$ are the roots of $g(z)$, then all roots of $h(z)$ are of the form $-w\alpha_j$ for some $j \in \{1, \dots, n\}$ and $w \in K$.

Chapter 2

Refinements of the Brunn-Minkowski inequality

As it was stated in the previous chapter, Brunn-Minkowski's theorem says that the function in $\lambda \in [0,1]$ given by $\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/n}$, where K, L are convex bodies, is concave. In this chapter we study refinements of the Brunn-Minkowski inequality, in the sense of 'enhancing' the exponent $1/n$, either when a common projection onto an $(n-k)$-plane is assumed or for particular families of sets. In the first case, we will show that the expected result of concavity for the k-th root of the volume is not true, although other Brunn-Minkowski type inequalities can be obtained under the $(n-k)$-projection hypothesis. In the second case, we show that for p-tangential bodies, the exponent in Brunn-Minkowski's inequality can be replaced by $1/p$. The original work that we collect in this chapter can be found in [31].

2.1 A counterexample for the concavity of the k-th root of the volume under a common $(n-k)$-projection assumption

In Section 50 of [9], linear refinements of the Brunn-Minkowski inequality are obtained for convex bodies having a common/equal-volume hyperplane projection (see also [46] for compact sets and more recently Subsection 1.2.4 of [20]).

Theorem 2.1.1 (Bonnesen, [9]; [20]). *Let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists a hyperplane $H \in \mathcal{L}^n_{n-1}$ with $K|H = L|H$. Then, for all $\lambda \in [0,1]$,*

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big) \geq (1-\lambda)\mathrm{vol}(K)+\lambda\mathrm{vol}(L). \tag{2.1}$$

This is, the volume itself is a concave function.

Theorem 2.1.2 (Bonnesen, [9]; [20, 46]). *Let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists a hyperplane $H \in \mathcal{L}^n_{n-1}$ with $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(L|H)$. Then, for all $\lambda \in [0,1]$,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)\mathrm{vol}(K) + \lambda\mathrm{vol}(L). \tag{2.2}$$

We would like to point out that, contrary to Theorem 2.1.1, Theorem 2.1.2 does not provide the concavity of the function $f(\lambda) = \mathrm{vol}\big((1-\lambda)K + \lambda L\big)$ for $\lambda \in [0,1]$. Indeed, inequality (2.2) only yields the condition $f(\lambda) \geq (1-\lambda)f(0) + \lambda f(1)$ and, in order to get the concavity of f one should be able to assure that

$$f\big((1-t)\lambda_1 + t\lambda_2\big) = \mathrm{vol}\Big((1-t)\big((1-\lambda_1)K + \lambda_1 L\big) + t\big((1-\lambda_2)K + \lambda_2 L\big)\Big) \geq (1-t)f(\lambda_1) + tf(\lambda_2).$$

Thus one should (be under the suitable conditions to) apply (2.2) to the bodies $(1-\lambda_1)K + \lambda_1 L$ and $(1-\lambda_2)K + \lambda_2 L$. However, in general, these sets do not have a common volume projection although the condition $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(L|H)$ holds. Moreover, there exists a counterexample in the literature which shows that the above-mentioned function is not concave under the sole assumption of common volume projection. For further details we refer to Notes for Section 7.7 in [52] and the references therein. Nevertheless, when working with convex bodies K and L with a common projection onto a hyperplane, this problem does not exist (see the proof of Theorem 2.1.3).

Regarding Theorem 2.1.1, Schneider proved in a very elegant way that even the most general Brunn-Minkowski inequality for mixed volumes (1.24) (and thus, in particular, the Brunn-Minkowski inequality for quermassintegrals (1.23)) admits an improved version of this type, unifying different results in the literature about this topic ([51], see also Section 7.7 of [52]): if $K, L \in \mathcal{K}^n$ are convex bodies such that there exists a hyperplane $H \in \mathcal{L}^n_{n-1}$ with $K|H = L|H$, then any mixed volume (and hence, quermassintegrals, volume) $\mathrm{V}\big((1-\lambda)K + \lambda L[m], K_{m+1}, \dots, K_n\big)$ itself of the convex combination $(1-\lambda)K + \lambda L$, for $K_{m+1}, \dots, K_n \in \mathcal{K}^n$, is a concave function in $\lambda \in [0,1]$.

At this point it is a natural question whether an analogous result to Theorem 2.1.1, but with the suitable exponent, can be obtained if a *common projection onto an $(n-k)$-dimensional plane* is assumed. Thus, the following property would be a natural expected solution:

> *Let $k \in \{1, \dots, n\}$ and let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists $H \in \mathcal{L}^n_{n-k}$ with $K|H = L|H$. Then for all $\lambda \in [0,1]$*
>
> $$\mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/k} \geq (1-\lambda)\mathrm{vol}(K)^{1/k} + \lambda\mathrm{vol}(L)^{1/k}.$$
>
> (2.3)

Here we show that this statement is not true; it is the content of Theorem 2.1.3. To this aim, we start by showing a couple of preliminary results which will be needed for its proof.

Lemma 2.1.1. *The first derivative of the function $\eta : (0, \infty) \longrightarrow \mathbb{R}$ given by*

$$\eta(x) = \sum_{k=0}^{\infty}\left(\left(x + k + \frac{1}{2}\right)\log\left(1 + \frac{1}{x+k}\right) - 1\right) \tag{2.4}$$

is concave.

Proof. Denoting by

$$\eta_k(x) = \left(x + k + \frac{1}{2}\right) \log\left(1 + \frac{1}{x+k}\right) - 1,$$

a straightforward computation shows that

$$\eta_k'''(x) = -\frac{2x + 2k + 1}{(x+k)^3(x+k+1)^3} < 0 \tag{2.5}$$

for all $x > 0$. Moreover, $\left|\eta_k'''(x)\right|$ is a decreasing function on $(0, \infty)$ and so

$$\left|\eta_k'''(x)\right| = \frac{2x + 2k + 1}{(x+k)^3(x+k+1)^3} \leq \frac{2k+1}{k^3(k+1)^3}. \tag{2.6}$$

Hence, since the numerical series

$$\sum_{k=0}^{\infty} \frac{2k+1}{k^3(k+1)^3} < \infty,$$

and by means of (2.6), the Weierstrass M-test ensures that $\sum_{k=0}^{\infty} \eta_k'''(x)$ converges uniformly on every $[a, b] \subset (0, \infty)$. Comparing with the series $\sum_{k=0}^{\infty} 1/k^2$, it is easy to check the convergence of

$$\sum_{k=0}^{\infty} \eta_k''(1), \qquad \sum_{k=0}^{\infty} \eta_k'(1) \quad \text{and} \quad \sum_{k=0}^{\infty} \eta_k(1),$$

and thus, by Theorem 1.1.1, we may assure that $\sum_{k=0}^{\infty} \eta_k''(x)$, $\sum_{k=0}^{\infty} \eta_k'(x)$ and $\sum_{k=0}^{\infty} \eta_k(x)$ converge uniformly on every $[a, b] \subset (0, \infty)$ with $1 \in [a, b]$. Furthermore, $\eta'(x) = \sum_{k=0}^{\infty} \eta_k'(x)$ for all $x > 0$, and so it is a concave function on $(0, \infty)$ (cf. (2.5)). □

Lemma 2.1.2 ([31]). *The sequence* $\left(\kappa_n \kappa_{n-2}/\kappa_{n-1}^2\right)_{n \geq 2}$ *is strictly increasing and*

$$\lim_{n \to \infty} \frac{\kappa_n \kappa_{n-2}}{\kappa_{n-1}^2} = 1.$$

Proof. On the one hand, we consider the real functions $f_i : (0, \infty) \longrightarrow \mathbb{R}$, $i = 1, 2$, given by $f_1(x) = (x - 1/2) \log x$ and $f_2(x) = \eta(x)$ where $\eta(x)$ is the function by (2.4). From the concavity of their first derivatives (cf. Lemma 2.1.1) we get

$$2f_i'\left(x + \frac{1}{2}\right) - f_i'(x) - f_i'(x+1) = 2\left[f_i'\left(x + \frac{1}{2}\right) - \frac{f_i'(x) + f_i'(x+1)}{2}\right] > 0,$$

and hence, the real functions $h_i : (0, \infty) \longrightarrow \mathbb{R}$, $i = 1, 2$, given by

$$h_i(x) = 2f_i\left(x + \frac{1}{2}\right) - f_i(x) - f_i(x+1)$$

are strictly increasing. Therefore, $e^{h_1(x) + h_2(x)}$ is also strictly increasing.

On the other hand, Stirling's formula (1.17) for the gamma function $\Gamma(x)$ allows to write

$$\frac{\Gamma\left(x + \frac{1}{2}\right)^2}{\Gamma(x)\Gamma(x+1)} = e^{h_1(x) + h_2(x)}.$$

Thus, all together, and using (1.15), we can conclude that

$$\frac{\kappa_n\kappa_{n-2}}{\kappa_{n-1}^2}=\frac{\Gamma\left(\frac{n-1}{2}+1\right)^2}{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n-2}{2}+1\right)}=e^{h_1\left(\frac{n}{2}\right)+h_2\left(\frac{n}{2}\right)}$$

is strictly increasing in n. The last assertion comes from the fact that

$$\lim_{n\to\infty}\frac{\kappa_{n-k}/\kappa_n}{\left(\kappa_{n-1}/\kappa_n\right)^k}=1$$

for all $k\geq 0$, and then, for $k=2$ we get the required result. Indeed, Stirling's formula (1.18) for the gamma function together with (1.15) yield the asymptotic formula

$$\lim_{n\to\infty}\frac{\kappa_n}{\left(\frac{2\pi e}{n}\right)^{n/2}\frac{1}{\sqrt{n\pi}}}=1.$$

Therefore we get

$$\begin{aligned}\lim_{n\to\infty}\frac{\kappa_{n-k}/\kappa_n}{\left(\kappa_{n-1}/\kappa_n\right)^k}&=\lim_{n\to\infty}\frac{\left(\frac{2\pi e}{n-k}\right)^{\frac{n-k}{2}}\frac{1}{\sqrt{(n-k)\pi}}}{\left(\frac{2\pi e}{n}\right)^{n/2}\frac{1}{\sqrt{n\pi}}}\left(\frac{\left(\frac{2\pi e}{n}\right)^{n/2}\frac{1}{\sqrt{n\pi}}}{\left(\frac{2\pi e}{n-1}\right)^{\frac{n-1}{2}}\frac{1}{\sqrt{(n-1)\pi}}}\right)^k\\&=\lim_{n\to\infty}\frac{(n-1)^{k/2}\sqrt{n}}{n^{k/2}\sqrt{n-k}}\,\frac{(n-1)^{(n-1)k/2}\,n^{n/2}}{(n-k)^{(n-k)/2}\,n^{nk/2}}=1.\end{aligned}$$

We point out that this fact will be shown in a more general setting in Lemma 4.2.3 of this dissertation; we have included here the proof of this particular case for the sake of completeness. □

In order to show that the statement (2.3) is, in general, not true, we explicitly construct the convex bodies providing a counterexample for it.

Theorem 2.1.3 ([31]). *For every $n\geq 3$, there exist convex bodies $K,L\in\mathcal{K}^n$, with a common $(n-2)$-dimensional projection $K|H=L|H$, $H\in\mathcal{L}^n_{n-2}$, such that, for all $\lambda\in(0,1)$,*

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/2}<(1-\lambda)\mathrm{vol}(K)^{1/2}+\lambda\mathrm{vol}(L)^{1/2}. \tag{2.7}$$

Proof. Let $g(\lambda)=\mathrm{vol}\big((1-\lambda)K+\lambda L\big)$ and $f(\lambda)=g(\lambda)^{1/2}$. On the one hand, we observe that the reverse inequality to (2.7), namely,

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/2}\geq(1-\lambda)\mathrm{vol}(K)^{1/2}+\lambda\mathrm{vol}(L)^{1/2}$$

for convex bodies $K,L\in\mathcal{K}^n$ having a common $(n-2)$-dimensional projection, $K|H=L|H$, holds if and only if $f(\lambda)$ is a concave function on $[0,1]$. Indeed, since $K|H=L|H$, then

$$(1-\lambda_1)K|H+\lambda_1L|H=(1-\lambda_2)K|H+\lambda_2L|H\quad\text{for any }\lambda_1,\lambda_2\in[0,1],$$

and thus the above inequality can be applied to the convex bodies $(1-\lambda_1)K+\lambda_1 L$, $(1-\lambda_2)K+\lambda_2 L$ in order to get the inequality $f\big((1-t)\lambda_1+t\lambda_2\big)\geq(1-t)f(\lambda_1)+tf(\lambda_2)$. Conversely, if f is a concave function on $[0,1]$ then we have, in particular, that $f(t)\geq(1-t)f(0)+tf(1)$, which gives the required inequality for the volume.

On the other hand, $f(\lambda)$ is concave if and only if

$$f''(\lambda)=\frac{1}{2}\,g(\lambda)^{-3/2}\left[g(\lambda)g''(\lambda)-\frac{1}{2}\,g'(\lambda)^2\right]\leq 0,$$

i.e., if and only if

$$F(\lambda)=g(\lambda)g''(\lambda)-\frac{1}{2}\,g'(\lambda)^2\leq 0.$$

Therefore, if we find two convex bodies $K,L\in\mathcal{K}^n$, having a common $(n-2)$-dimensional projection, and satisfying that $F(\lambda)>0$ for all $\lambda\in[0,1]$, then inequality (2.7) will hold for all $\lambda\in(0,1)$.

Let $L=B_n$ and $K=M+B_n$, with $M\in\mathcal{K}^2_0$ lying in a 2-dimensional linear plane (see Figure 2.1). On the one hand, it is clear that if $H=(\operatorname{lin}M)^\perp\in\mathcal{L}^n_{n-2}$ is the orthogonal complement of $\operatorname{lin}M$, then $K|H=B_n|H$.

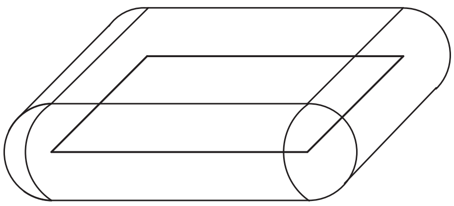

Figure 2.1: The counterexample: $M+B_3$ and B_3, with $\dim M=2$.

On the other hand, Steiner formula (1.8) allows us to write

$$g(\lambda)=\operatorname{vol}\big((1-\lambda)(M+B_n)+\lambda B_n\big)=\operatorname{vol}\big((1-\lambda)M+B_n\big)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(M)(1-\lambda)^{n-i},$$

and since $\dim M=2$, the quermassintegrals $\mathrm{W}_i(M)$ take the values

$$\mathrm{W}_i(M)=0,\quad i=0,\dots,n-3,$$
$$\mathrm{W}_{n-2}(M)=\frac{2\kappa_{n-2}}{n(n-1)}\mathrm{A}(M),\qquad \mathrm{W}_{n-1}(M)=\frac{\kappa_{n-1}}{2n}\mathrm{p}(M)$$

(cf. (1.4)), and therefore,

$$\begin{aligned}g(\lambda)&=\frac{n(n-1)}{2}\mathrm{W}_{n-2}(M)(1-\lambda)^2+n\mathrm{W}_{n-1}(M)(1-\lambda)+\kappa_n\\&=\kappa_{n-2}\mathrm{A}(M)(1-\lambda)^2+\frac{\kappa_{n-1}}{2}\mathrm{p}(M)(1-\lambda)+\kappa_n.\end{aligned}$$

Thus, it is an easy computation to check that

$$F(\lambda) = g(\lambda)g''(\lambda) - \frac{g'(\lambda)^2}{2} = \frac{1}{8}\Big[16\kappa_n\kappa_{n-2}\mathrm{A}(M) - \kappa_{n-1}^2\mathrm{p}(M)^2\Big],$$

and does not depend on λ. So, $F(\lambda) > 0$ if and only if there exists a planar convex body M satisfying that

$$\mathrm{p}(M)^2 < 16\frac{\kappa_n\kappa_{n-2}}{\kappa_{n-1}^2}\mathrm{A}(M) \tag{2.8}$$

for all $n \geq 3$. We observe that $\kappa_n\kappa_{n-2}/\kappa_{n-1}^2$ is strictly increasing for $n \geq 2$ (see Lemma 2.1.2), and hence, since $n \geq 3$, we have

$$16\frac{\kappa_n\kappa_{n-2}}{\kappa_{n-1}^2} > 16\frac{\kappa_2\kappa_0}{\kappa_1^2} = 4\pi = \frac{\mathrm{p}(B_2)^2}{\mathrm{A}(B_2)}.$$

Thus, the planar unit ball B_2 satisfies (2.8) for any value of the dimension. It finishes the proof. In fact, many planar convex bodies satisfy (2.8). □

An analogous argument also shows that the corresponding expected refinement for the Brunn-Minkowski inequality for quermassintegrals (1.23) (when $E = B_n$), namely,

$$\mathrm{W}_i\big((1-\lambda)K + \lambda L\big)^{1/k} \geq (1-\lambda)\mathrm{W}_i(K)^{1/k} + \lambda\mathrm{W}_i(L)^{1/k}, \tag{2.9}$$

is not possible:

Proposition 2.1.1 ([31]). *Let $i \in \mathbb{N}$ be fixed. Then there exists $n_0 \geq i+3$ such that, for all $n \geq n_0$, there are convex bodies $K, L \in \mathcal{K}^n$, with a common $(n-2)$-dimensional projection, satisfying that, for all $\lambda \in (0,1)$,*

$$\mathrm{W}_i\big((1-\lambda)K + \lambda L\big)^{1/2} < (1-\lambda)\mathrm{W}_i(K)^{1/2} + \lambda\mathrm{W}_i(L)^{1/2}. \tag{2.10}$$

Proof. Let $g(\lambda) = \mathrm{W}_i\big((1-\lambda)K+\lambda L\big)$ and $f(\lambda) = g(\lambda)^{1/2}$. Arguing in the same way as in the proof of Theorem 2.1.3, we conclude that if we find two convex bodies $K, L \in \mathcal{K}^n$, having a common $(n-2)$-dimensional projection, n large enough, and satisfying that $F(\lambda) = g(\lambda)g''(\lambda) - (1/2)g'(\lambda)^2 > 0$ for all $\lambda \in [0,1]$, then inequality (2.10) will hold for all $\lambda \in (0,1)$.

Again, let $L = B_n$ and $K = M + B_n$ with $\dim M = 2$, for which the projection condition is fulfilled. Similar computations as before show that $F(\lambda) > 0$ if and only if there exists a planar convex body M satisfying that

$$\mathrm{p}(M)^2 < 16\frac{n(n-i-1)}{(n-1)(n-i)}\frac{\kappa_n\kappa_{n-2}}{\kappa_{n-1}^2}\mathrm{A}(M). \tag{2.11}$$

It is easy to check that the function $n(n-i-1)/\big((n-1)(n-i)\big)$ is strictly increasing in n if $n \geq (i+1)/2$ (in particular, for $n \geq i+3$) for fixed i, and has limit 1 when n goes to infinity. Since

$\kappa_n\kappa_{n-2}/\kappa_{n-1}^2$ is also increasing in the dimension and tends to 1 when $n \to \infty$ (see Lemma 2.1.2), the product of both functions is increasing and we get

$$\lim_{n\to\infty} 16\frac{n(n-i-1)}{(n-1)(n-i)}\frac{\kappa_n\kappa_{n-2}}{\kappa_{n-1}^2} = 16 > 4\pi = \frac{\mathrm{p}(B_2)^2}{\mathrm{A}(B_2)}.$$

Thus, if $n_0 \in \mathbb{N}$ is the first value of the dimension such that the planar unit ball B_2 satisfies (2.11) (the above condition for the limit ensures that n_0 always exists), then the monotonicity shows that for all $n \geq n_0$, inequality (2.10) holds for $K = B_2 + B_n$ and $L = B_n$. □

Remark 2.1. *We observe, for instance, that in the case $i = 1$, the value of the dimension from which inequality* (2.10) *holds is $n_0 = 5$.*

In Section 50 of [9] a similar result to Theorem 2.1.2, but involving sections instead of projections, is mentioned (a proof can be found in Corollary 1.2.1 of [20] for $\lambda = 1/2$):

Theorem 2.1.4 (Bonnesen, [9]; [20]). *If*

$$\max_{x\in H^\perp} \mathrm{vol}_{n-1}\bigl(K \cap (H + x)\bigr) = \max_{x\in H^\perp} \mathrm{vol}_{n-1}\bigl(L \cap (H + x)\bigr), \tag{2.12}$$

for $K, L \in \mathcal{K}^n$ and some hyperplane $H \in \mathcal{L}^n_{n-1}$, then for all $\lambda \in [0, 1]$

$$\mathrm{vol}\bigl((1 - \lambda)K + \lambda L\bigr) \geq (1 - \lambda)\mathrm{vol}(K) + \lambda\mathrm{vol}(L).$$

The same construction can be made in order to show that an analogous result for $(n - k)$-dimensional sections will be not true: indeed, since the convex bodies $K = B_2 + B_n$ and $L = B_n$ are symmetric with respect to the origin, for any $(n-2)$-plane H, the section $K\cap(H+x)$, $x \in H^\perp$, with maximum $(n - 2)$-dimensional volume is the one through the origin, i.e., $K \cap H$, which coincides with $K|H$ (analogously for $L = B_n$). Therefore, choosing H as in the proof of Theorem 2.1.3, condition (2.12) is fulfilled, but we get that $\mathrm{vol}\bigl((1/2)(K+L)\bigr)^{1/2} < (1/2)\mathrm{vol}(K)^{1/2}+(1/2)\mathrm{vol}(L)^{1/2}$. Moreover, the same inequality is obtained when working with any $\lambda \in (0, 1)$.

2.2 Refinements of the Brunn-Minkowski inequality involving projections

In the previous section we have shown that statement (2.3) is not true (Theorem 2.1.3). Therefore, either additional assumptions should be imposed in order to get such a result or, under that precise hypothesis, a different inequality can be obtained. In this sense, we get Propositions 2.2.1 and 2.2.2. In order to state some of these results, we need the following extra notation, which will be used throughout all the chapter: given $K \in \mathcal{K}^n$ and $H \in \mathcal{L}^n_{n-k}$, we will write, for any $u \in K|H$,

$$K(u) = \left\{x \in \mathbb{R}^k : \begin{pmatrix} x \\ u \end{pmatrix} \in K\right\}. \tag{2.13}$$

The proofs of these results follow the idea of the proofs of Theorems 2.1.1 and 2.1.2 in [20].

Proposition 2.2.1 ([31]). *Let $k \in \{1,\dots,n\}$ and let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists $H \in \mathcal{L}^n_{n-k}$ with $K|H = L|H$. Then, for all $\lambda \in [0,1]$,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)^k \mathrm{vol}(K) + \lambda^k \mathrm{vol}(L). \tag{2.14}$$

Proof. Without loss of generality we assume that $H = \big\{(x_1,\dots,x_n)^\intercal \in \mathbb{R}^n : x_1 = \dots = x_k = 0\big\}$, and for the sake of brevity we write, on the one hand,

$$U = K|H = L|H \quad \text{and} \quad M_\lambda = (1-\lambda)K + \lambda L.$$

Thus, $M_\lambda|H = (1-\lambda)(K|H) + \lambda(L|H) = U$, for all $\lambda \in [0,1]$. On the other hand, it is clear that for all $u \in U$ and any $x \in K(u)$, $y \in L(u)$ (cf. (2.13)), we have

$$\begin{pmatrix} (1-\lambda)x + \lambda y \\ u \end{pmatrix} = (1-\lambda)\begin{pmatrix} x \\ u \end{pmatrix} + \lambda \begin{pmatrix} y \\ u \end{pmatrix} \in M_\lambda,$$

and therefore, $(1-\lambda)K(u) + \lambda L(u) \subset M_\lambda(u)$. Hence, using Fubini's Theorem 1.2.1 and Brunn-Minkowski's inequality (1.22), we get

$$\begin{aligned} \mathrm{vol}\big((1-\lambda)K + \lambda L\big) = \mathrm{vol}(M_\lambda) &= \int_U \mathrm{vol}_k\big(M_\lambda(u)\big)\,\mathrm{d}u \geq \int_U \mathrm{vol}_k\big((1-\lambda)K(u) + \lambda L(u)\big)\,\mathrm{d}u \\ &\geq \int_U \Big((1-\lambda)\mathrm{vol}_k\big(K(u)\big)^{1/k} + \lambda \mathrm{vol}_k\big(L(u)\big)^{1/k}\Big)^k \mathrm{d}u \\ &\geq \int_U \Big((1-\lambda)^k \mathrm{vol}_k\big(K(u)\big) + \lambda^k \mathrm{vol}_k\big(L(u)\big)\Big)\,\mathrm{d}u \\ &= (1-\lambda)^k \mathrm{vol}(K) + \lambda^k \mathrm{vol}(L). \end{aligned}$$

□

As in the case of Theorems 2.1.1 and 2.1.2, the same inequality (2.14) can be obtained when the identity assumption on projections is weakened to a condition between $(n-k)$-dimensional volumes.

Proposition 2.2.2 ([31]). *Let $k \in \{1,\dots,n\}$ and let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists $H \in \mathcal{L}^n_{n-k}$ with $\mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H)$. Then, for all $\lambda \in [0,1]$,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)^k \mathrm{vol}(K) + \lambda^k \mathrm{vol}(L).$$

Proof. Applying Schwarz symmetrization to the convex bodies K, L and $(1-\lambda)K+\lambda L$, with respect to the $(n-k)$-plane H, yields new convex bodies $K' = \sigma_H(K)$, $L' = \sigma_H(L)$ and $\sigma_H\big((1-\lambda)K + \lambda L\big)$ satisfying

$$(1-\lambda)K' + \lambda L' \subset \sigma_H\big((1-\lambda)K + \lambda L\big)$$

(see Lemma 1.3.1 ii)), and since Schwarz symmetrization preserves the volume (Lemma 1.3.1 i)), it suffices to prove that

$$\mathrm{vol}\big((1-\lambda)K' + \lambda L'\big) \geq (1-\lambda)^k \mathrm{vol}(K') + \lambda^k \mathrm{vol}(L').$$

Next, we notice that $K|H = K' \cap H$ and, moreover,

$$\mathrm{vol}_{n-k}\big(K' \cap H\big) = \max_{t \in H^\perp} \mathrm{vol}_{n-k}\big(K' \cap (t + H)\big),$$

and analogously for the convex body L. Then, applying again Schwarz symmetrization to the sets K', L', but now with respect to $H^\perp$, we get new convex bodies $\sigma_{H^\perp}(K'), \sigma_{H^\perp}(L')$ satisfying that

$$\big(\sigma_{H^\perp}(K')\big)\,|H = \left(\frac{\mathrm{vol}_{n-k}(K|H)}{\kappa_{n-k}}\right)^{1/(n-k)} B_{n-k},$$
$$\big(\sigma_{H^\perp}(L')\big)\,|H = \left(\frac{\mathrm{vol}_{n-k}(L|H)}{\kappa_{n-k}}\right)^{1/(n-k)} B_{n-k},$$

and since $\mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H)$, we obtain that

$$\big(\sigma_{H^\perp}(K')\big)\,|H = \big(\sigma_{H^\perp}(L')\big)\,|H.$$

Thus, we can apply Proposition 2.2.1 to the convex bodies $\sigma_{H^\perp}(K'), \sigma_{H^\perp}(L')$ which, together with the facts that the volume is preserved and the inclusion

$$(1-\lambda)\sigma_{H^\perp}(K') + \lambda\,\sigma_{H^\perp}(L') \subset \sigma_{H^\perp}\big((1-\lambda)K' + \lambda L'\big)$$

holds, yields

$$\begin{aligned}\mathrm{vol}\big((1-\lambda)K' + \lambda L'\big) &= \mathrm{vol}\Big(\sigma_{H^\perp}\big((1-\lambda)K' + \lambda L'\big)\Big) \geq \mathrm{vol}\Big((1-\lambda)\sigma_{H^\perp}(K') + \lambda\,\sigma_{H^\perp}(L')\Big)\\ &\geq (1-\lambda)^k \mathrm{vol}\big(\sigma_{H^\perp}(K')\big) + \lambda^k \mathrm{vol}\big(\sigma_{H^\perp}(L')\big)\\ &= (1-\lambda)^k \mathrm{vol}(K') + \lambda^k \mathrm{vol}(L').\end{aligned}$$
□

Remark 2.2. *We observe that Brunn-Minkowski's inequality* (1.22) *implies that*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)^n \mathrm{vol}(K) + \lambda^n \mathrm{vol}(L).$$

Therefore inequality (2.14) *generalizes the above one for $k = n$ and* (2.1) *when $k = 1$.*

Inequality (2.14) can be also obtained if we replace the property about the projection volume $\mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H)$ by a section volume condition:

Proposition 2.2.3 ([31]). *Let $k \in \{1, \ldots, n\}$ and let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists $H \in \mathcal{L}^n_{n-k}$ with*

$$\max_{x \in H^\perp} \mathrm{vol}_{n-k}\big(K \cap (x + H)\big) = \max_{x \in H^\perp} \mathrm{vol}_{n-k}\big(L \cap (x + H)\big).$$

Then, for all $\lambda \in [0,1]$,

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)^k \mathrm{vol}(K) + \lambda^k \mathrm{vol}(L).$$

This result generalizes Theorem 2.1.4 to all $k \in \{1, \dots, n\}$. The proof is a direct consequence of Proposition 2.2.2 and the following lemma.

Lemma 2.2.1 ([31]). *Let $k \in \{1, \dots, n\}$ and $H \in \mathcal{L}^n_{n-k}$. The following statements are equivalent:*

i) If $K, L \in \mathcal{K}^n$ satisfy that $\mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H)$, then inequality (2.14) holds for all $\lambda \in [0,1]$.

ii) If $K, L \in \mathcal{K}^n$ satisfy that $\max_{x \in H^\perp} \mathrm{vol}_{n-k}\big(K \cap (x+H)\big) = \max_{x \in H^\perp} \mathrm{vol}_{n-k}\big(L \cap (x+H)\big)$, then inequality (2.14) holds for all $\lambda \in [0,1]$.

Proof. First, we suppose i) and assume that

$$\max_{x \in H^\perp} \mathrm{vol}_{n-k}\big(K \cap (x+H)\big) = \max_{x \in H^\perp} \mathrm{vol}_{n-k}\big(L \cap (x+H)\big) = \nu.$$

Then the orthogonal projections onto H of the Schwarz symmetrals of K and L with respect to $H^\perp$, namely, $\sigma_{H^\perp}(K)$, $\sigma_{H^\perp}(L)$, are equal; more precisely,

$$\big(\sigma_{H^\perp}(K)\big)|H = \left(\frac{\nu}{\kappa_{n-k}}\right)^{1/(n-k)} B_{n-k} = \big(\sigma_{H^\perp}(L)\big)|H.$$

Thus part i), together with the known properties of the Schwarz symmetrization (see Lemma 1.3.1), allows to conclude that

$$\begin{aligned}
\mathrm{vol}\big((1-\lambda)K + \lambda L\big) &= \mathrm{vol}\Big(\sigma_{H^\perp}\big((1-\lambda)K + \lambda L\big)\Big) \geq \mathrm{vol}\big((1-\lambda)\sigma_{H^\perp}(K) + \lambda\, \sigma_{H^\perp}(L)\big) \\
&\geq (1-\lambda)^k \mathrm{vol}\big(\sigma_{H^\perp}(K)\big) + \lambda^k \mathrm{vol}\big(\sigma_{H^\perp}(L)\big) \\
&= (1-\lambda)^k \mathrm{vol}(K) + \lambda^k \mathrm{vol}(L).
\end{aligned}$$

Conversely, we now suppose ii) and assume that $\mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H)$. Then the Schwarz symmetrals $\sigma_H(K)$ and $\sigma_H(L)$ satisfy that

$$\begin{aligned}
\max_{x \in H^\perp} \mathrm{vol}_{n-k}\big((\sigma_H(K)) \cap (x+H)\big) &= \mathrm{vol}_{n-k}\big((\sigma_H(K)) \cap H\big) = \mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H) \\
&= \max_{x \in H^\perp} \mathrm{vol}_{n-k}\big((\sigma_H(L)) \cap (x+H)\big),
\end{aligned}$$

and therefore, ii), together with known properties of the Schwarz symmetrization, yields

$$\begin{aligned}
\mathrm{vol}\big((1-\lambda)K + \lambda L\big) &= \mathrm{vol}\Big(\sigma_H\big((1-\lambda)K + \lambda L\big)\Big) \geq \mathrm{vol}\big((1-\lambda)\sigma_H(K) + \lambda\, \sigma_H(L)\big) \\
&\geq (1-\lambda)^k \mathrm{vol}(\sigma_H(K)) + \lambda^k \mathrm{vol}(\sigma_H(L)) \\
&= (1-\lambda)^k \mathrm{vol}(K) + \lambda^k \mathrm{vol}(L).
\end{aligned}$$

□

We observe that the above relation (2.14) has inequality (2.1) as a particular case; however, Brunn-Minkowski's inequality cannot be obtained from it (see Remark 2.2). Next theorem provides an extension of both inequalities (2.14) and (1.22) (see Remark 2.3).

Theorem 2.2.1 ([31]). *Let $k \in \{1, \dots, n\}$ and let $K, L \in \mathcal{K}^n$ be convex bodies such that there exists $H \in \mathcal{L}^n_{n-k}$ with $K|H = L|H = U$. Then, for all $\lambda \in [0,1]$,*

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/k} \geq (1-\lambda)\int_U \left(\frac{\mathrm{vol}_k\big(K(u)\big)}{\mathrm{vol}_{n-k}(U)^{k-1}}\right)^{1/k} \mathrm{d}u + \lambda \int_U \left(\frac{\mathrm{vol}_k\big(L(u)\big)}{\mathrm{vol}_{n-k}(U)^{k-1}}\right)^{1/k} \mathrm{d}u. \quad (2.15)$$

Proof. Arguing as in the proof of Proposition 2.2.1, we get $(1-\lambda)K(u) + \lambda L(u) \subset M_\lambda(u)$ which, together with Brunn-Minkowski's inequality (1.22), yields

$$\begin{aligned}\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/k} &= \mathrm{vol}(M_\lambda)^{1/k} = \left(\int_U \mathrm{vol}_k\big(M_\lambda(u)\big)\,\mathrm{d}u\right)^{1/k}\\ &\geq \left(\int_U \mathrm{vol}_k\big((1-\lambda)K(u)+\lambda L(u)\big)\,\mathrm{d}u\right)^{1/k}\\ &\geq \left(\int_U \Big[(1-\lambda)\mathrm{vol}_k\big(K(u)\big)^{1/k} + \lambda \mathrm{vol}_k\big(L(u)\big)^{1/k}\Big]^k \mathrm{d}u\right)^{1/k}.\end{aligned}$$

Then, applying Hölder's inequality (1.28) to the functions $(1-\lambda)\mathrm{vol}_k\big(K(u)\big)^{1/k} + \lambda \mathrm{vol}_k\big(L(u)\big)^{1/k}$ and 1, we finally get

$$\begin{aligned}\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/k} &\geq \left(\int_U \Big[(1-\lambda)\mathrm{vol}_k\big(K(u)\big)^{1/k} + \lambda \mathrm{vol}_k\big(L(u)\big)^{1/k}\Big]^k \mathrm{d}u\right)^{1/k}\\ &\geq \frac{1}{\mathrm{vol}_{n-k}(U)^{1-1/k}} \int_U \Big[(1-\lambda)\mathrm{vol}_k\big(K(u)\big)^{1/k} + \lambda \mathrm{vol}_k\big(L(u)\big)^{1/k}\Big]\,\mathrm{d}u\\ &= (1-\lambda)\int_U \left(\frac{\mathrm{vol}_k\big(K(u)\big)}{\mathrm{vol}_{n-k}(U)^{k-1}}\right)^{1/k} \mathrm{d}u + \lambda \int_U \left(\frac{\mathrm{vol}_k\big(L(u)\big)}{\mathrm{vol}_{n-k}(U)^{k-1}}\right)^{1/k} \mathrm{d}u. \quad \square\end{aligned}$$

Remark 2.3. *Theorem 2.2.1 generalizes both, Theorem 2.1.1 and Brunn-Minkowski's inequality* (1.22)*. Indeed, if $k = 1$ then* (2.15) *becomes* (2.1)*; for $k = n$, then $U = \{0\}$ and hence,* $\mathrm{vol}_0(U) = 1$*, and the integrals in* (2.15) *are just the volumes of K and L, respectively. Thus,* (2.15) *gives* (1.22).

We conclude this section by showing that, for a particular relative quermassintegral, the expected refinement can be obtained (cf. (2.9)).

Proposition 2.2.4 ([31]). *Let $k \in \{1, \dots, n-1\}$ and let $K, L \in \mathcal{K}^n_0$ be such that there exists $H \in \mathcal{L}^n_{n-k}$ with $K|H = L|H$. Then, for any convex body $E_k \subsetneq H^\perp$, $\dim E_k = k$, and all $\lambda \in [0,1]$,*

$$\mathrm{W}_{k-1}\big((1-\lambda)K+\lambda L; E_k\big)^{1/k} \geq (1-\lambda)\mathrm{W}_{k-1}(K;E_k)^{1/k} + \lambda \mathrm{W}_{k-1}(L;E_k)^{1/k}. \quad (2.16)$$

Proof. We observe that (2.16) holds for convex bodies K, L having a common $(n-k)$-projection if and only if $f(\lambda) = \mathrm{W}_{k-1}\big((1-\lambda)K+\lambda L; E_k\big)^{1/k}$ is a concave function on $[0,1]$ (see the proof of Theorem 2.1.3). So we have to see that $f''(\lambda) \leq 0$, and following the argument of the proof of Theorem 7.4.5 in [52], it suffices to show this for $\lambda = 0$: indeed, if $0 < \lambda < 1$ we set

$$K_\tau = (1-\tau)\big((1-\lambda)K+\lambda L\big) + \tau L \quad \text{and} \quad h(\tau) = \mathrm{W}_{k-1}(K_\tau; E_k)^{1/k},$$

$\tau \in [0,1]$; since $f(\lambda+\rho) = h\big(\rho/(1-\lambda)\big)$, $0 \leq \rho \leq 1-\lambda$, then $f''(\lambda) \leq 0$ follows from $h''(0) \leq 0$. Now, using (1.3) it can be checked that $f''(0) \leq 0$ if and only if

$$(n-k)\mathrm{W}_{k-1}(K;E_k)\Big[\mathrm{W}_{k-1}(K;E_k) - 2\mathrm{V}\big(K[n-k],L,E_k[k-1]\big) + \mathrm{V}\big(K[n-k-1],L[2],E_k[k-1]\big)\Big] - \left(1-\frac{1}{k}\right)(n-k+1)\Big[\mathrm{V}\big(K[n-k],L,E_k[k-1]\big) - \mathrm{W}_{k-1}(K;E_k)\Big]^2 \leq 0.$$

The second summand is clearly negative, and hence, we have to study the sign of the first one.

On the one hand, denoting for short $\mathcal{C} = \big(K[n-k-1], E_k[k-1]\big)$, it is well-known that

$$\frac{\mathrm{V}(K[2],\mathcal{C})}{\mathrm{V}(K,M,\mathcal{C})^2} - \frac{2\mathrm{V}(K,L,\mathcal{C})}{\mathrm{V}(K,M,\mathcal{C})\mathrm{V}(L,M,\mathcal{C})} + \frac{\mathrm{V}(L[2],\mathcal{C})}{\mathrm{V}(L,M,\mathcal{C})^2} \leq 0 \tag{2.17}$$

for any convex body $M \in \mathcal{K}^n$ such that the above mixed volumes are not zero (see Theorem 7.4.3 in [52]). On the other hand, since $E_k \subsetneq H^\perp$, we have (see Theorem 1.2.3)

$$\mathrm{W}_k(K;E_k) = \frac{1}{\binom{n}{k}}\mathrm{vol}_k(E_k)\mathrm{vol}_{n-k}(K|H) \qquad \text{and}$$

$$\mathrm{V}\big(K[n-k-1],L,E_k[k]\big) = \mathrm{vol}_k(E_k)\mathrm{V}\left(K[n-k-1],L,\frac{E_k}{\mathrm{vol}_k(E_k)^{1/k}}[k]\right)$$
$$= \frac{\mathrm{vol}_k(E_k)}{\binom{n}{k}}\vartheta^{(n-k)}\big(K|H[n-k-1],L|H\big),$$

and using the projection assumption $K|H = L|H$, we get

$$\mathrm{V}\big(K[n-k-1],L,E_k[k]\big) = \frac{\mathrm{vol}_k(E_k)}{\binom{n}{k}}\mathrm{vol}_{n-k}(K|H) = \mathrm{W}_k(K;E_k),$$

i.e., $\mathrm{V}(L,E_k,\mathcal{C}) = \mathrm{V}(K,E_k,\mathcal{C})$. Then, (2.17) for $M = E_k$ yields

$$\mathrm{W}_{k-1}(K;E_k) - 2\mathrm{V}\big(K[n-k],L,E_k[k-1]\big) + \mathrm{V}\big(K[n-k-1],L[2],E_k[k-1]\big) \leq 0,$$

which shows that $f''(0) \leq 0$, as required. □

2.3 Brunn-Minkowski type inequalities for particular families of convex bodies

In this section, we wonder whether refinements of Brunn-Minkowski's inequality of type (2.3) can be obtained for particular families of sets or under additional assumptions. First we show that, among others, it has a positive answer for the family of the so-called p-tangential bodies (see Definition 1.11). In this case, also a refinement of the more general Brunn-Minkowski inequality for quermassintegrals (1.23) can be achieved (see Theorem 2.3.1). We start by showing the following theorem, which is an improvement of the Brunn-Minkowski inequality for the family of p-tangential bodies, and which is a direct consequence of a slightly more general result (Theorem 2.3.2).

Theorem 2.3.1 ([31]). *Let K be a p-tangential body of $E \in \mathcal{K}_0^n$, $1 \le p \le n-1$. Then*

$$\mathrm{vol}\big((1-\lambda)K + \lambda E\big)^{1/p} \ge (1-\lambda)\mathrm{vol}(K)^{1/p} + \lambda \mathrm{vol}(E)^{1/p}$$

for all $\lambda \in [0,1]$, and equality holds if and only if $K = E$. Moreover,

$$\mathrm{W}_i\big((1-\lambda)K + \lambda E; K\big)^{1/(p-i)} \ge (1-\lambda)\mathrm{W}_i(K;K)^{1/(p-i)} + \lambda \mathrm{W}_i(E;K)^{1/(p-i)},$$

$i = 0, \dots, p-1$, and equality holds for a fixed i, if and only if K is also an i-tangential body of E.

Next result shows that the above inequality holds even in the case when the inclusion $E \subset K$ does not hold (for any translate of E); roughly speaking, it is sufficient that "first" quermassintegrals are equal (cf. Theorem 1.3.1), a condition which is satisfied by a class of convex bodies bigger than tangential bodies.

Theorem 2.3.2 ([31]). *Let $K \in \mathcal{K}^n$, $E \in \mathcal{K}_0^n$ be convex bodies and $s \in \{1, \dots, n\}$ such that $\mathrm{W}_s(K;E) = \mathrm{W}_{s+1}(K;E) = \cdots = \mathrm{W}_n(K;E)$. Then, for all $\lambda \in [0,1]$,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda E\big)^{1/s} \ge (1-\lambda)\mathrm{vol}(K)^{1/s} + \lambda \mathrm{vol}(E)^{1/s}, \tag{2.18}$$

and equality holds if and only if $K = E$. Moreover,

$$\mathrm{W}_i\big((1-\lambda)K + \lambda E; E\big)^{1/(s-i)} \ge (1-\lambda)\mathrm{W}_i(K;E)^{1/(s-i)} + \lambda \mathrm{W}_i(E;E)^{1/(s-i)},$$

$i = 0, \dots, s-1$, and equality holds for a fixed i, if and only if $\mathrm{W}_i(K;E) = \cdots = \mathrm{W}_n(K;E)$.

Before showing this result, we see how Theorem 2.3.1 can be deduced from it: if K is a p-tangential body of $E \in \mathcal{K}_0^n$, $1 \le p \le n-1$, then Favard's Theorem 1.3.1 ensures that

$$\mathrm{W}_0(K;E) = \mathrm{W}_1(K;E) = \cdots = \mathrm{W}_{n-p}(K;E) \ne 0,$$

and since $\mathrm{W}_j(K;E) = \mathrm{W}_{n-j}(E;K)$, Theorem 2.3.2 immediately implies Theorem 2.3.1.

Now, we deal with the proof of Theorem 2.3.2.

Proof. We will show the inequality (2.18) for the volume. The relations for the quermassintegrals, as well the corresponding equality cases, can be obtained analogously.

Using the well-known Aleksandrov-Fenchel inequality for quermassintegrals (1.20) and since $\mathrm{W}_s(K;E) \ne 0$, we easily get that

$$\mathrm{W}_0(K;E) \le \cdots \le \mathrm{W}_{s-1}(K;E) \le \mathrm{W}_s(K;E) = \cdots = \mathrm{W}_n(K;E). \tag{2.19}$$

Now we consider the polynomial function

$$f(\lambda) = \sum_{i=0}^{s} \binom{s}{i} \mathrm{W}_i(K;E)(1-\lambda)^{s-i}\lambda^i,$$

for $\lambda \in [0,1]$. On the one hand, we can write

$$f(\lambda) = \bigl[(1-\lambda)+\lambda\bigr]^{n-s} f(\lambda) = \left(\sum_{j=0}^{n-s}\binom{n-s}{j}(1-\lambda)^{n-s-j}\lambda^j\right)\left(\sum_{i=0}^{s}\binom{s}{i}\mathrm{W}_i(K;E)(1-\lambda)^{s-i}\lambda^i\right)$$
$$= \sum_{k=0}^{n}\left(\sum_{i+j=k}\mathrm{W}_i(K;E)\binom{s}{i}\binom{n-s}{j}\right)(1-\lambda)^{n-k}\lambda^k,$$

and using (2.19) we get that

$$\sum_{i+j=k}\mathrm{W}_i(K;E)\binom{s}{i}\binom{n-s}{j} \le \mathrm{W}_k(K;E)\sum_{i+j=k}\binom{s}{i}\binom{n-s}{j} = \mathrm{W}_k(K;E)\binom{n}{k}.$$

Therefore,

$$f(\lambda)^{1/s} \le \left(\sum_{k=0}^{n}\binom{n}{k}\mathrm{W}_k(K;E)(1-\lambda)^{n-k}\lambda^k\right)^{1/s} = \mathrm{vol}\bigl((1-\lambda)K+\lambda E\bigr)^{1/s}. \tag{2.20}$$

On the other hand, since the coefficients of the polynomial $f(\lambda)$, namely, $\mathrm{W}_i(K;E)$ for $i=0,\dots,s$, are non-negative real numbers satisfying the Aleksandrov-Fenchel inequalities (1.20), Theorem 1.4.2 ensures that $\mathrm{W}_i(K;E) = \mathrm{W}_i^{(s)}(K_s;E_s)$ are the relative quermassintegrals of two convex bodies $K_s, E_s \in \mathcal{K}^s$, $i=0,\dots,s$. Then, using (2.20), Brunn-Minkowski's inequality (1.22) in $\mathbb{R}^s$, and since

$$\mathrm{vol}_s(K_s) = \mathrm{W}_0^{(s)}(K_s;E_s) = \mathrm{W}_0(K;E) \qquad \text{and}$$
$$\mathrm{vol}_s(E_s) = \mathrm{W}_s^{(s)}(K_s;E_s) = \mathrm{W}_s(K;E) = \mathrm{W}_n(K;E),$$

we can conclude that

$$\mathrm{vol}\bigl((1-\lambda)K+\lambda E\bigr)^{1/s} \ge f(\lambda)^{1/s} = \mathrm{vol}_s\bigl((1-\lambda)K_s+\lambda E_s\bigr)^{1/s}$$
$$\ge (1-\lambda)\mathrm{vol}_s(K_s)^{1/s} + \lambda\mathrm{vol}_s(E_s)^{1/s} = (1-\lambda)\mathrm{vol}(K)^{1/s} + \lambda\mathrm{vol}(E)^{1/s}.$$

If $K=E$ then equality holds in (2.18). Conversely, if we have equality in (2.18), then equality holds in (2.19) for all quermassintegrals, i.e., $\mathrm{W}_0(K;E) = \cdots = \mathrm{W}_n(K;E)$. It implies that $K=E$. □

Remark 2.4. *The condition* $\mathrm{int}\, E \neq \emptyset$ *cannot be removed, since it is needed that* $\mathrm{vol}(E) \neq 0$. *Indeed, taking* $K = [0,\mathrm{e}_1]+[0,\mathrm{e}_2]+[0,2\mathrm{e}_3] \in \mathcal{K}^3$ *and* $E = [0,\mathrm{e}_3]$, *then* $\mathrm{W}_2(K;E) = \mathrm{W}_3(K;E) = 0$. *However, for every* $\lambda \in (0,1)$,

$$\mathrm{vol}\bigl((1-\lambda)K+\lambda L\bigr)^{1/2} = (1-\lambda)(2-\lambda)^{1/2} < (1-\lambda)\sqrt{2} = (1-\lambda)\mathrm{vol}(K)^{1/2} + \lambda\mathrm{vol}(E)^{1/2}.$$

In order to conclude the chapter, we make an observation regarding another family of convex bodies for which a refinement of Brunn-Minkowski's inequality can be obtained, namely, the family $\mathcal{V} = \bigl\{K \in \mathcal{K}^n : \mathrm{vol}(K) = \nu\bigr\}$, for a fixed positive real number $\nu \in \mathbb{R}_{>0}$: if $K, L \in \mathcal{V}$, then the multiplicative version of Brunn-Minkowski's inequality (Theorem 1.4.6) leads to

$$\mathrm{vol}\bigl((1-\lambda)K+\lambda L\bigr) \ge \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda} = \nu = (1-\lambda)\,\mathrm{vol}(K) + \lambda\mathrm{vol}(L).$$

Thus, the following corollary has been proved.

Corollary 2.3.1 ([31]). *Let* $K, L \in \mathcal{K}^n$ *with* $\mathrm{vol}(K) = \mathrm{vol}(L)$*. Then,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)\mathrm{vol}(K) + \lambda\mathrm{vol}(L).$$

The above result can be also obtained as a consequence (for $k = 0$) of a more general refinement of the Brunn-Minkowski type inequality (2.3) for quermassintegrals.

Proposition 2.3.1 ([31]). *Let* $k \in \{0, \ldots, n-2\}$ *and let* $K, L \in \mathcal{K}^n$ *be convex bodies such that*

$$\mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H) \quad \textit{for all } \; H \in \mathcal{L}^n_{n-k}. \tag{2.21}$$

Then,

$$\mathrm{W}_k\big((1-\lambda)K + \lambda L\big) \geq (1-\lambda)\mathrm{W}_k(K) + \lambda\mathrm{W}_k(L).$$

Proof. Kubota's integral recursion formula (see e.g. identity (5.72) in [52]) states, in particular, that, for any convex body $K \in \mathcal{K}^n$,

$$\mathrm{W}_k(K) = \frac{\kappa_n}{\kappa_{n-k}} \int_{\mathcal{L}^n_{n-k}} \mathrm{vol}_{n-k}(K|H)\,\mathrm{d}\mu(H),$$

where μ is the (rotationally invariant) Haar measure on the set $\mathcal{L}^n_{n-k}$ such that $\mu(\mathcal{L}^n_{n-k}) = 1$. Thus, since $\mathrm{vol}_{n-k}(K|H) = \mathrm{vol}_{n-k}(L|H)$, we immediately get that $\mathrm{W}_k(K) = \mathrm{W}_k(L)$, and moreover, using Brunn-Minkowski's inequality in $\mathbb{R}^k$ we can conclude that

$$\begin{aligned}
\int_{\mathcal{L}^n_{n-k}} \mathrm{vol}_{n-k}\Big(\big((1-\lambda)K + \lambda L\big)\big|H\Big)\,\mathrm{d}\mu(H) &= \int_{\mathcal{L}^n_{n-k}} \mathrm{vol}_{n-k}\big((1-\lambda)K|H + \lambda L|H\big)\,\mathrm{d}\mu(H) \\
&\geq \int_{\mathcal{L}^n_{n-k}} \Big[(1-\lambda)\mathrm{vol}_{n-k}(K|H)^{1/(n-k)} + \lambda\mathrm{vol}_{n-k}(L|H)^{1/(n-k)}\Big]^{n-k} \mathrm{d}\mu(H) \\
&= \int_{\mathcal{L}^n_{n-k}} \mathrm{vol}_{n-k}(K|H)\,\mathrm{d}\mu(H).
\end{aligned}$$

Therefore,

$$\mathrm{W}_k\big((1-\lambda)K + \lambda L\big) \geq \mathrm{W}_k(K) = (1-\lambda)\mathrm{W}_k(K) + \lambda\mathrm{W}_k(L). \qquad \square$$

We would like to point out that the assumption (2.21) is not so restrictive: it is known in the literature that if $K|H = L|H$ for all $H \in \mathcal{L}^n_{n-k}$, then $K = L$; moreover, Aleksandrov's projection theorem (see e.g. Corollary 8.1.5 in [52]) states that if $K, L \in \mathcal{K}^n_0$ are 0-symmetric convex bodies such that $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(L|H)$ for all $H \in \mathcal{L}^n_{n-1}$, then (up to translations) $K = L$. However, there exist convex bodies $K \neq L$ satisfying (2.21), i.e., Aleksandrov's projection theorem is not true neither for non-symmetric convex bodies nor for projections onto $(n-k)$-dimensional planes, with $k > 1$.

Chapter 3

Linearity of the volume. Looking for a characterization of sausages

Let $\mathrm{V}_{K;E}(\lambda) = \mathrm{vol}\big(\lambda K + (1-\lambda)E\big)$ be the volume of the convex combination of $K, E \in \mathcal{K}^n$ for $\lambda \in [0,1]$ which, for convenience, will be written $\lambda K + (1-\lambda)E$ throughout this chapter. From (1.5) it follows that $\mathrm{V}_{K;E}(\lambda)$ is a polynomial of degree (at most) n, namely,

$$\mathrm{V}_{K;E}(\lambda) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)\lambda^{n-i}(1-\lambda)^i.$$

Brunn-Minkowski's inequality (1.22) ensures that the function $\mathrm{V}_{K;E}^{1/n}$ defined on $\lambda \in [0,1]$ is concave. In the previous chapter, we have studied that under special assumptions on the convex bodies K, E relative to a projection onto a hyperplane (Theorems 2.1.1 and 2.1.2) the classical Brunn-Minkowski inequality can be refined obtaining that $\mathrm{V}_{K;E}(\lambda) \geq \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E)$.

In this chapter, we prove that under the sole assumption that K and E have an equal volume projection (or a common maximal volume section through parallel hyperplanes to a given one), if equality holds in the above inequality for just one value λ in $(0,1)$, then (up to degenerated convex bodies) the pair K, E is a sausage, i.e., we characterize the equality case of Theorem 2.1.2 (and thus also Theorem 2.1.1) and Theorem 2.1.4. However, even having equality for all $\lambda \in [0,1]$, if no extra assumption on K, E is done, such a characterization is not possible. This problem is connected with a conjecture relating the roots of the Steiner polynomial of a pair of convex bodies and their relative inradius; counterexamples for the general case are explicitly given. In the same line, a counterexample to a conjecture by Matheron of 1978 on inner parallel bodies is also shown. The original work that we collect in this chapter can be found in [50, 62].

3.1 Linearity of the volume and the sausage conjecture: counterexamples

In [26], the following statement was conjectured (see Definition 1.12):

Conjecture 3.1.1. *Let $K\in\mathcal{K}^n$ with inradius $\mathrm{r}(K)=1$. Then -1 is an $(n-1)$-fold root of $f_{K;B_n}(z)$ if and only if K is a sausage with respect to B_n.*

The classical Bonnesen inequality in the plane establishes that

$$\mathrm{A}(K)-\mathrm{p}(K)\mathrm{r}(K)+\kappa_2\mathrm{r}(K)^2\leq 0,$$

with equality if and only if $K=L+\mathrm{r}(K)B_n$ with $L\in\mathcal{K}^2$, $\dim L\leq 1$. This was proved by Bonnesen in [8]. Then Blaschke generalized it to an arbitrary gauge body E in the plane (pages 33-36 in [7]):

$$\mathrm{W}_0(K;E)-2\mathrm{W}_1(K;E)\mathrm{r}(K;E)+\mathrm{W}_2(K;E)\mathrm{r}(K;E)^2\leq 0. \tag{3.1}$$

Again, equality holds if and only if $K=L+\mathrm{r}(K;E)E$ with $L\in\mathcal{K}^2$, $\dim L\leq 1$. Thus, Conjecture 3.1.1 is true in dimension 2 for any gauge body E.

In this section we intend to understand/characterize the (pairs of) convex bodies K,E for which $\mathrm{V}_{K;E}$ is a linear function in $\lambda\in[0,1]$, i.e., those bodies for which

$$\mathrm{V}_{K;E}(\lambda)=\lambda\mathrm{vol}(K)+(1-\lambda)\mathrm{vol}(E) \tag{3.2}$$

for all $\lambda\in[0,1]$. From now on, whenever we refer to the linearity of the volume we will mean (3.2). First, we will prove in the following result that both, linearity of the volume and Conjecture 3.1.1, are closely related: indeed, (3.2) is equivalent to the fact that -1 is an $(n-1)$-fold root of $f_{K;E}(z)$. It will come from an expression for the (relative) i-th quermassintegral of $\lambda K+(1-\lambda)E$ involving the derivatives of the Steiner polynomial $f_{K;E}(z)$ (cf. (1.6)). We notice that the j-th derivative of $f_{K;E}(z)$ is given by

$$f_{K;E}^{(j)}(z)=j!\sum_{i=j}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\binom{i}{j}z^{i-j}.$$

From now on and for the sake of brevity, we will write $K_\lambda=\lambda K+(1-\lambda)E$ for $\lambda\in[0,1]$.

Lemma 3.1.1 ([50]). *Let $K,E\in\mathcal{K}^n$ be convex bodies. Then, for $i=0,\dots,n$*

$$\mathrm{W}_i\big(\lambda K+(1-\lambda)E;E\big)=\frac{1}{\binom{n}{i}}\sum_{j=0}^{n-i}\binom{n-j}{i}\frac{f_{K;E}^{(n-j)}(-1)}{(n-j)!}\lambda^j.$$

Proof. Using the linearity of mixed volumes, we can write the quermassintegrals $\mathrm{W}_i(K_\lambda;E)$ for $i=0,\dots,n-1$ as polynomials in λ (cf. also Theorem 1.2.5):

$$\mathrm{W}_i(K_\lambda;E)=\sum_{k=0}^{n-i}\binom{n-i}{k}\mathrm{W}_{i+k}(K;E)\lambda^{n-(i+k)}(1-\lambda)^k.$$

By rearranging the terms we obtain that

$$\begin{aligned}
\mathrm{W}_i(K_\lambda;E) &= \sum_{k=0}^{n-i}\binom{n-i}{k}\mathrm{W}_{i+k}(K;E)\lambda^{n-(i+k)}(1-\lambda)^k\\
&= \sum_{k=0}^{n-i}\binom{n-i}{k}\mathrm{W}_{i+k}(K;E)\sum_{j=0}^{k}\binom{k}{j}(-1)^j\lambda^{n-i-(k-j)}\\
&= \sum_{l=0}^{n-i}\left(\sum_{k=l}^{n-i}\binom{n-i}{k}\mathrm{W}_{i+k}(K;E)\binom{k}{l}(-1)^{k-l}\right)\lambda^{n-i-l}\\
&= \sum_{l=0}^{n-i}\left(\sum_{k=l}^{n-i}\frac{\binom{n}{i+k}\binom{i+k}{i}}{\binom{n}{i}}\mathrm{W}_{i+k}(K;E)\binom{k}{l}(-1)^{k-l}\right)\lambda^{n-i-l}\\
&= \frac{1}{\binom{n}{i}}\sum_{j=i}^{n}\left(\sum_{r=j}^{n}\binom{n}{r}\mathrm{W}_r(K;E)\binom{r-i}{j-i}\binom{r}{i}(-1)^{r-j}\right)\lambda^{n-j}\\
&= \frac{1}{\binom{n}{i}}\sum_{j=i}^{n}\left(\binom{j}{i}\sum_{r=j}^{n}\binom{n}{r}\mathrm{W}_r(K;E)\binom{r}{j}(-1)^{r-j}\right)\lambda^{n-j}\\
&= \frac{1}{\binom{n}{i}}\sum_{j=i}^{n}\binom{j}{i}\frac{f_{K;E}^{(j)}(-1)}{j!}\lambda^{n-j}.
\end{aligned}$$

□

In particular, for $i=0$, we have

$$\mathrm{vol}\big(\lambda K+(1-\lambda)E\big)=\sum_{j=0}^{n}\frac{f_{K;E}^{(n-j)}(-1)}{(n-j)!}\lambda^j, \tag{3.3}$$

and hence, from the above result, we immediately get the announced equivalence:

Corollary 3.1.1 ([50]). *For $K,E\in\mathcal{K}^n$, $\mathrm{V}_{K;E}(\lambda)$ is linear if and only if -1 is an $(n-1)$-fold root of $f_{K;E}(z)$. In this case, we also have linearity for all quermassintegrals $\mathrm{W}_i(K_\lambda;E)$, $i=1,\dots,n$.*

Remark 3.1. *From Lemma 3.1.1 we know that if $\mathrm{W}_{i_0}(K_\lambda;E)$ is linear for some $i_0\in\{0,\dots,n-2\}$, then $\mathrm{W}_i(K_\lambda;E)$ is also linear for all $i>i_0$. The converse is not true, as the following example shows. For $n=2$ there is nothing to see, because W_1 is always linear. For $n=3$, the numbers $\mathrm{W}_0=9$, $\mathrm{W}_1=7$, $\mathrm{W}_2=4$ and $\mathrm{W}_3=1$, satisfy inequalities* (1.20) *and hence, Theorem 1.4.2 ensures that there exist $K,E\in\mathcal{K}^n$ such that $\mathrm{W}_i(K;E)=\mathrm{W}_i$, which yields $f_{K;E}(z)=9+21z+12z^2+z^3$. Thus $f_{K;E}(-1)=-1$, $f'_{K;E}(-1)=0$ and we have that $\mathrm{W}_i(K_\lambda;E)$ is linear for $i=1,2,3$, but $\mathrm{W}_0(K_\lambda;E)$ is not so. In higher dimensions similar examples can be constructed.*

Good candidates for (pairs of) convex bodies characterizing the linearity of the volume are the sausages: fixing a convex body E, let $K=L+E$, with $L\in\mathcal{K}^n$ so that $\dim L\leq 1$. Then, by (1.5), for these bodies we have

$$\mathrm{vol}\big(\lambda K+(1-\lambda)E\big)=\mathrm{vol}\big(\lambda L+E\big)=n\mathrm{W}_{n-1}(L;E)\lambda+\mathrm{vol}(E)=\lambda\mathrm{vol}(K)+(1-\lambda)\mathrm{vol}(E). \tag{3.4}$$

So, one might think that this family allows to characterize the linearity of the volume. In fact, considering full-dimensional convex bodies K, E having equal volume, the following remark ensures that, in this case, only 'degenerated' sausages, i.e., $K = L + E$ with $\dim L = 0$ can turn up.

Remark 3.2. *Let $K, E \in \mathcal{K}^n$ satisfying (3.2) for some $\lambda \in (0,1)$. The following facts hold:*

i) *If $\mathrm{vol}(K) = \mathrm{vol}(E)$ then*

$$\mathrm{V}_{K;E}(\lambda) = \mathrm{vol}(K) = \Big(\lambda \mathrm{vol}(K)^{1/n} + (1-\lambda)\mathrm{vol}(E)^{1/n}\Big)^n$$

and the equality case in Brunn-Minkowski's inequality yields that either K and E lie in parallel hyperplanes if $\dim K, \dim E < n$ or, since $\mathrm{vol}(K) = \mathrm{vol}(E)$, then $K = E$ (up to translations).

ii) *Conversely, if for some $\lambda \in (0,1)$*

$$\mathrm{V}_{K;E}(\lambda) = \Big(\lambda \mathrm{vol}(K)^{1/n} + (1-\lambda)\mathrm{vol}(E)^{1/n}\Big)^n,$$

then from the strict concavity of $(\cdot)^{1/n}$ it follows that either K and E lie in parallel hyperplanes or $K = E$ (up to translations).

In the following we will suppose, without loss of generality, that $\mathrm{vol}(K) \neq \mathrm{vol}(E)$. Despite all the traces, sausages are not the only (pairs of) convex bodies satisfying linearity of the volume as it is shown in the next proposition. They are, in turn, not so far from being the only ones, as it follows from Theorems 3.2.1 and 3.2.3. There, the sole additional assumption that the bodies have a common volume projection or a common maximum volume section provides a characterization for sausages.

Proposition 3.1.1 ([50]). *There exist convex bodies $K, E \in \mathcal{K}^n$, $n \geq 2$, such that K, E is not a sausage, and satisfying*

$$\mathrm{vol}\big(\lambda K + (1-\lambda)E\big) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E) \quad \textit{for all } \lambda \in [0,1].$$

Proof. Let $L \in \mathcal{K}^n$ with $\dim L = 1$. Obviously, the quermassintegrals $\mathrm{W}_i(L + B_n)$ of $L + B_n$ with respect to B_n, satisfy the inequalities (1.20) for $1 \leq i \leq n-1$, and hence, by Theorem 1.4.2 there exist simplices K and E such that $\mathrm{W}_i(K; E) = \mathrm{W}_i(L + B_n)$. Thus $f_{K;E}(z) = f_{L+B_n;B_n}(z)$ which, together with (3.4) yield the linearity of $\mathrm{V}_{K;E}(\lambda)$.

Finally, we notice that a simplex K is a sausage with respect to another simplex E if and only if they coincide (up to a translations), which cannot be the case because

$$\mathrm{vol}(K) = \mathrm{W}_0(L + B_n) = \mathrm{vol}(L + B_n) > \mathrm{vol}(B_n) = \mathrm{W}_n(L + B_n) = \mathrm{vol}(E). \qquad \square$$

The (pairs of) convex bodies for which $\mathrm{V}_{K;E}$ is linear, i.e., such that -1 is an $(n-1)$-fold root of $f_{K;E}(z)$ (see Corollary 3.1.1), satisfy also other properties, as showed in the next result.

Lemma 3.1.2 ([50]). *Let $K, E \in \mathcal{K}^n$. If -1 is an $(n-1)$-fold root of $f_{K;E}(z)$ then*

i) $\mathrm{W}_0(K;E) - \mathrm{W}_1(K;E) = \mathrm{W}_i(K;E) - \mathrm{W}_{i+1}(K;E)$ *for* $i = 0, \dots, n-1$.

ii) $f^{(i)}_{K_\lambda;E}(-1) = 0$ *for* $i = 0, \dots, n-2$, *and any* $\lambda \in [0,1]$.

Proof. If -1 is an $(n-1)$-fold root of $f_{K;E}(z)$, then it is also a root of its $(n-2)$-th derivative, i.e.,

$$0 = \mathrm{W}_{n-2}(K;E) - 2\mathrm{W}_{n-1}(K;E) + \mathrm{W}_n(K;E),$$

which can be read as $\mathrm{W}_{n-2}(K;E) - \mathrm{W}_{n-1}(K;E) = \mathrm{W}_{n-1}(K;E) - \mathrm{W}_n(K;E)$.

Now, we assume by reverse induction on $j \leq 0$ that

$$\mathrm{W}_s(K;E) - \mathrm{W}_{s+1}(K;E) = \mathrm{W}_{n-1}(K;E) - \mathrm{W}_n(K;E) \quad \text{for all } n-1 \geq s > j,$$

and we substitute this in the j-th derivative of $f_{K;E}(z)$. Then by arranging the terms we obtain

$$\begin{aligned}
0 = \frac{f^{(j)}_{K;E}(-1)}{\binom{n}{j} j!} &= \sum_{i=0}^{n-j} \binom{n-j}{i} \mathrm{W}_{j+i}(K;E)(-1)^i \\
&= \mathrm{W}_j(K;E) + \sum_{i=1}^{n-j-1} \left(\binom{n-j-1}{i-1} + \binom{n-j-1}{i} \right) \mathrm{W}_{j+i}(K;E)(-1)^i + \mathrm{W}_n(K;E)(-1)^{(n-j)} \\
&= \mathrm{W}_j(K;E) - \mathrm{W}_{j+1}(K;E) + \sum_{i=1}^{n-j-1} \binom{n-j-1}{i} \big(\mathrm{W}_{j+i}(K;E) - \mathrm{W}_{j+i+1}(K;E)\big)(-1)^i \\
&= \mathrm{W}_j(K;E) - \mathrm{W}_{j+1}(K;E) + \big(\mathrm{W}_{n-1}(K;E) - \mathrm{W}_n(K;E)\big) \sum_{i=1}^{n-j-1} \binom{n-j-1}{i} (-1)^i \\
&= \mathrm{W}_j(K;E) - \mathrm{W}_{j+1}(K;E) - \big(\mathrm{W}_{n-1}(K;E) - \mathrm{W}_n(K;E)\big),
\end{aligned}$$

which concludes the proof of i).

In order to prove the second assertion, we notice that, since -1 is an $(n-1)$-fold root of $f_{K;E}(z)$, we have $\mathrm{W}_i(K_\lambda;E) = \lambda \mathrm{W}_i(K;E) + (1-\lambda)\mathrm{W}_i(E;E)$ (see Corollary 3.1.1) and hence

$$\mathrm{W}_i(K_\lambda;E) - \mathrm{W}_{i+1}(K_\lambda;E) = \lambda\big(\mathrm{W}_i(K;E) - \mathrm{W}_{i+1}(K;E)\big) \quad \text{for } i = 0, \dots, n-1.$$

Therefore we have, by the previous item, that $\mathrm{W}_0(K_\lambda;E) - \mathrm{W}_1(K_\lambda;E) = \mathrm{W}_i(K_\lambda;E) - \mathrm{W}_{i+1}(K_\lambda;E)$ for $i = 0, \dots, n-1$, and substituting on successive derivatives of $f_{K_\lambda;E}(z)$ we obtain that, as it also happens for K, $f^{(j)}_{K_\lambda;E}(-1) = 0$, $j = 0, 1, \dots, n-2$. □

We would like to mention that under the assumption of a common projection of K and E, it is known (see Theorem 7.7.3 in [52]) that *i)* implies that the pair K, E is a sausage.

Indeed, this is a consequence of some results which support Conjecture 3.1.1. Its validity is known in some special cases where additional hypothesis, such as a common/equal-volume projection onto a hyperplane, are assumed. For completeness we collect the cases in which the validity of Conjecture 3.1.1 is known.

Remark 3.3. i) *If there exists a hyperplane* $H\in\mathcal{L}^n_{n-1}$ *so that* $\mathrm{W}^{(n-1)}_{n-2}(K|H;B_{n-1})=\kappa_{n-1}$*, or equivalently, if the mean width of* $K|H$*, in the ambient space* H*, coincides with the mean width of the unit ball in* H*, i.e., 2, and*

$$\mathrm{W}_{n-2}(K)-2\mathrm{W}_{n-1}(K)+\kappa_n=0,$$

then K *is the sum of a segment and the unit ball (see [12]).*

From this result it follows that if K *is a convex body having a common projection with the unit ball,* $K|H=B_{n-1}=B_n|H$*, then* -1 *is an* $(n-1)$*-fold root of* $f_{K;B_n}(z)$ *if and only if* K *is a sausage with respect to* B_n*.*

ii) *If there exists* $H\in\mathcal{L}^n_{n-1}$ *so that* $K|H=E|H$*, with* $\dim E|H=n-1$*, then Conjecture 3.1.1 follows from Theorem 7.7.3 in [52].*

iii) *The above two cases are closely related to Theorem 3.3 in [26], since this one can be obtained from them when the set of incenters of* K *is not a unique point. Indeed, let* K *have inradius* $\mathrm{r}(K)=1$*. If all 2-dimensional projections of* K *have (2-dimensional) inradius* 1*, the set of incenters of* K *is at most 1-dimensional; otherwise, some of the projections would have greater inradius. Since the set of incenters is not a singleton, there is at least a 1-dimensional (convex and compact) set of incenters* l*. Furthermore if there exists a point* $p\in K$*,* $p\notin(\mathrm{aff}\,l)+B_n$*, then* $\mathrm{conv}\bigl(p\cup(l+B_n)\bigr)|\,\mathrm{aff\,conv}\bigl(l\cup\{p\}\bigr)$ *has inradius greater than* 1*, a contradiction. So* K *has an* $(n-1)$*-dimensional projection being an* $(n-1)$*-unit ball.*

Lemma 3.1.2 and the comments afterwards, together with the previous remark and the fact that $\mathrm{r}(K_\lambda;E)=\mathrm{r}(K;E)=1$ might lead to think that Conjecture 3.1.1 is true (for every gauge body E) as it occurs in dimension 2 (cf. (3.1)). The following result shows that this statement is not true: we explicitly construct convex bodies providing a counterexample.

Theorem 3.1.1 ([50]). *For* $n=3$*, there exist convex bodies* $K,E\in\mathcal{K}^3$*, with* $-\mathrm{r}(K;E)$ *being an* $(n-1)$*-fold root of* $f_{K;E}(z)$*, such that* K,E *is not a sausage.*

Proof. Embedding the unit cube C_2 in the plane $\bigl\{(x_1,x_2,x_3)^\intercal\in\mathbb{R}^3:\,x_3=0\bigr\}$, let

$$\widetilde{C}_2=\left(\frac{1}{2},\frac{1}{2},0\right)^\intercal+C_2$$

and let $L=[0,\mathrm{e}_3]$. We take $\overline{C}_1=\bigl[(0,1,1)^\intercal,(1,1,1)^\intercal\bigr]$. Now, for $\tau\in[0,1]$ fixed, we define by $A_\tau=[0,\mathrm{e}_1]+\tau[0,\mathrm{e}_2]\subset\widetilde{C}_2$ the orthogonal box of edge-lengths 1 and τ.

Let $L_1=\bigl[(0,\tau,\tau)^\intercal,(1,\tau,\tau)^\intercal\bigr]$ be the segment, parallel to $[0,\mathrm{e}_1]$ lying in the diagonal face $\mathrm{conv}\left([0,\mathrm{e}_1]\cup\overline{C}_1\right)$ of the 3-dimensional unit cube $(1/2,1/2,1/2)^\intercal+C_3$, whose projection onto the plane $\bigl\{(x_1,x_2,x_3)^\intercal\in\mathbb{R}^3:\,x_3=0\bigr\}$ is the edge $\bigl[(0,\tau,0)^\intercal,(1,\tau,0)^\intercal\bigr]$ of A_τ.

Finally, we consider

$$K = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^{\intercal} + C_3 = L + \widetilde{C}_2 \quad \text{and} \quad E = \operatorname{conv}(A_\tau \cup L_1)$$

the triangular prism determined by L_1 and A_τ (see Figure 3.1).

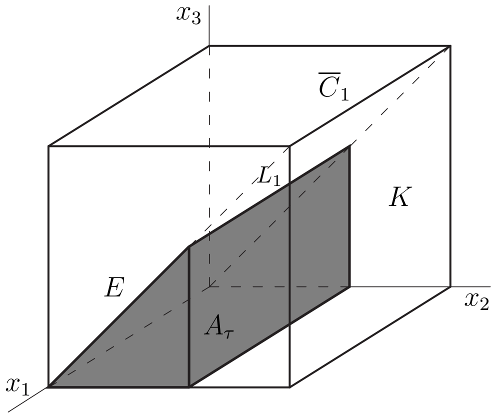

Figure 3.1: The counterexample

Then, on the one hand, it is clear that $\mathrm{r}(K;E) = 1$. On the other hand, for $\lambda \geq 0$, and denoting by $M(s)$ the section of $M \in \mathcal{K}^3$ with the plane $\{(x_1, x_2, x_3)^\intercal \in \mathbb{R}^3 : x_3 = s\}$, we have

$$\begin{aligned} f_{K;E}(\lambda) &= \operatorname{vol}(K + \lambda E) = \operatorname{vol}\big(L + \widetilde{C}_2 + \lambda E\big) = \operatorname{vol}_2\big((\widetilde{C}_2 + \lambda E)|L^\perp\big) + \operatorname{vol}\big(\widetilde{C}_2 + \lambda E\big) \\ &= \operatorname{vol}_2\big(\widetilde{C}_2 + \lambda A_\tau\big) + \int_0^{\lambda\tau} \operatorname{vol}_2\big((\widetilde{C}_2 + \lambda E)(s)\big)\,\mathrm{d}s \\ &= (\lambda + 1)(\lambda\tau + 1) + \int_0^{\lambda\tau} \operatorname{vol}_2\big((\widetilde{C}_2 + \lambda E)(s)\big)\,\mathrm{d}s. \end{aligned}$$

Since

$$\begin{aligned} \int_0^{\lambda\tau} \operatorname{vol}_2\big((\widetilde{C}_2 + \lambda E)(s)\big)\,\mathrm{d}s &= \int_0^{\lambda\tau} \operatorname{vol}_2\left(\left(1 - \frac{s}{\lambda\tau}\right)\lambda A_\tau + \frac{s}{\lambda\tau}\lambda L_1 + \widetilde{C}_2\right)\mathrm{d}s \\ &= \lambda\tau \int_0^1 \operatorname{vol}_2\left((1-t)\lambda A_\tau + t\lambda L_1 + \widetilde{C}_2\right)\,\mathrm{d}t \\ &= \lambda\tau\left(\int_0^1 \operatorname{vol}_2\big((1-t)\lambda A_\tau + \widetilde{C}_2\big)\,\mathrm{d}t + \int_0^1 t\lambda \operatorname{vol}_1\Big(\big((1-t)\lambda A_\tau + \widetilde{C}_2\big)|L_1^\perp\Big)\mathrm{d}t\right) \\ &= \lambda\tau \int_0^1 \big((1-t)\lambda\tau + 1\big)\big((1-t)\lambda + 1 + t\lambda\big)\,\mathrm{d}t \\ &= \lambda\tau(\lambda + 1)\left(1 + \lambda\tau \int_0^1 (1-t)\,\mathrm{d}t\right) \\ &= \lambda\tau(\lambda + 1)\left(\frac{\lambda\tau}{2} + 1\right), \end{aligned}$$

we have

$$f_{K;E}(\lambda) = \frac{1}{2}(\lambda+1)\left(\lambda^2\tau^2 + 4\lambda\tau + 2\right).$$

Finally, since $\tau^2 - 4\tau + 2 = 0$ if and only if $\tau = 2 \pm \sqrt{2}$, if we take $\tau = 2 - \sqrt{2} \in [0,1]$, then we have that $-1 = -\mathrm{r}(K;E)$ is a 2-fold root of $f_{K;E}(z)$. However, it is clear that K is not a sausage with respect to E, which concludes the proof. □

We have been not able to extend the above construction to the n-dimensional case. Nevertheless, if degenerated gauge bodies E are considered, a pair of convex bodies $K, E \in \mathcal{K}^n$ providing a counterexample can be obtained as follows:

Remark 3.4. *Following the same notation as in the proof of Theorem 3.1.1, let*

$$K = \left(\frac{1}{2},\dots,\frac{1}{2}\right)^{\intercal} + C_n = L + \widetilde{C}_{n-1}$$

be the unit cube and let

$$E = \operatorname{conv}\left\{\widetilde{C}_{n-2}, \frac{1}{2}\left(\widetilde{C}_{n-2} + \overline{C}_{n-2}\right)\right\}$$

be the diagonal 'half-face' of the cube K determined by $\widetilde{C}_{n-2}$. It is clear that K is not a sausage with respect to E and $\mathrm{r}(K;E) = 1$. However we have

$$\begin{aligned} f_{K;E}(\lambda) &= \mathrm{vol}(K+\lambda E) = \mathrm{vol}\left(L + \widetilde{C}_{n-1} + \lambda E\right) = \mathrm{vol}\left(\widetilde{C}_{n-1} + \lambda E\right) + \mathrm{vol}_{n-1}\left(\left(\widetilde{C}_{n-1} + \lambda E\right)|L^{\perp}\right) \\ &= \frac{\lambda}{2}(\lambda+1)^{(n-2)} + \left(\frac{\lambda}{2}+1\right)(\lambda+1)^{n-2} = (\lambda+1)^{n-1}. \end{aligned}$$

To the best of our knowledge it is not known whether for some other fixed gauge body E, in particular for the Euclidean ball B_n, Conjecture 3.1.1 holds true. In fact, the problem of classifying the gauge bodies E, if there are any, for which Conjecture 3.1.1 is true for any K remains open. So far we only know that they are not the whole $\mathcal{K}^n$, as the above results show.

Nevertheless, known results (see Remark 3.3) ensure the validity of Conjecture 3.1.1 in some special cases where an additional hypothesis, such as a common projection onto a hyperplane, is assumed.

This fact suggests that one may get a characterization of the linearity of the volume, under the additional assumption of a common projection onto a hyperplane. We have been able to characterize the convex bodies for which the volume function is linear even with a weaker assumption, namely, that the convex bodies have an equal volume hyperplane projection. This problem, as well as other related questions, will be one of the aims of the next section. Before showing it, in the following subsection, we deal with a conjecture by Matheron closely related to Conjecture 3.1.1; we will show how a slight modification of the bodies that provided us the above counterexample can be also used in order to disprove such conjecture.

3.1.1 A counterexample to a conjecture by Matheron

For two convex bodies $K, E \in \mathcal{K}^n$ with interior points and $0 \leq \lambda \leq \mathrm{r}(K;E)$, the *inner parallel body of K (relative to E) at distance λ* is the set

$$K \sim \lambda E = \{x \in \mathbb{R}^n : x + \lambda E \subset K\}.$$

It is easy to check that if $\mathrm{r}(K;E)E$ is a *summand* of K, i.e., if there exists $L \in \mathcal{K}^n$ such that $K = L + \mathrm{r}(K;E)E$, then (see e.g. [41] and page 225 of [52])

$$\mathrm{W}_i(K \sim \lambda E; E) = \sum_{k=0}^{n-i} \binom{n-i}{k} \mathrm{W}_{i+k}(K;E)(-\lambda)^k \tag{3.5}$$

for $0 \leq \lambda \leq \mathrm{r}(K;E)$ and $i = 0, \dots, n$. In [41] Matheron proved that the validity of (3.5) for $0 < \lambda < \mathrm{r}(K;E)$ and all $i = 0, \dots, n$ implies that $\mathrm{r}(K;E)E$ is a summand of K. He conjectured that it was enough to assume (3.5) just for $i = 0$:

Conjecture 3.1.2 (Matheron, [41]). *Let $K, E \in \mathcal{K}^n$ be convex bodies with interior points. Then*

$$\mathrm{vol}(K \sim \lambda E) \geq \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)(-\lambda)^i \tag{3.6}$$

for all $0 < \lambda < \mathrm{r}(K;E)$ with equality if and only if $\mathrm{r}(K;E)E$ is a summand of K.

The right hand side in (3.6) is usually called the *alternating Steiner polynomial* of K with respect to E. Matheron proved Conjecture 3.1.2 for $n = 2$.

In [28] it is proven that it is not possible to bound the volume of $K \sim \lambda E$ in terms of just the alternating Steiner polynomial. So, the counterexample(s) to the Matheron conjecture contained in [28] shows that the inequality part of the conjecture is not true. However, the equality case of this conjecture have not been considered yet, i.e., it was open whether there exist convex bodies K, E satisfying that $\mathrm{vol}(K \sim \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)(-\lambda)^i$, and so that the pair K, E is not a sausage. In the next theorem, an answer to this question is provided: the convex bodies K, E given in Theorem 3.1.2 are not a sausage; however the above condition is fulfilled.

Theorem 3.1.2 ([50]). *There exist convex bodies $K, E \in \mathcal{K}^3$ with interior points satisfying*

$$\mathrm{vol}(K \sim \lambda E) = \sum_{i=0}^{3} \binom{3}{i} \mathrm{W}_i(K;E)(-\lambda)^i$$

for all $0 < \lambda < \mathrm{r}(K;E)$ and such that $\mathrm{r}(K;E)E$ is not a summand of K.

Proof. Following the same notation as in the proof of Theorem 3.1.1, we take the orthogonal box $A = (1/4)[0, \mathrm{e}_1] + (3/4)[0, \mathrm{e}_2] \subset \widetilde{C}_2$ of edge-lengths $1/4$ and $3/4$, and let

$$L_1 = \left[\left(0, \frac{3}{4}, \frac{3}{4}\right)^{\intercal}, \left(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right)^{\intercal}\right]$$

be the segment (of length 1/4) parallel to $[0, \mathrm{e}_1]$ lying in the diagonal face $\operatorname{conv}\bigl([0, \mathrm{e}_1] \cup \overline{C}_1\bigr)$ of the unit cube $(1/2, 1/2, 1/2)^\intercal + C_3$.

Thus, if we consider (see Figure 3.1)

$$K = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^\intercal + C_3 = L + \widetilde{C}_2 \quad \text{and} \quad E = \operatorname{conv}(A \cup L_1)$$

the triangular prism determined by L_1 and A, it is easy to check, on the one hand, that

$$\operatorname{vol}(K \sim \lambda E) = \left(1 - \frac{3}{4}\lambda\right)^2 \left(1 - \frac{1}{4}\lambda\right), \quad \text{for all } 0 \le \lambda \le \frac{4}{3} = \mathrm{r}(K; E).$$

On the other hand, a similar computation as in the proof of Theorem 3.1.1 shows that, for $\lambda \ge 0$,

$$f_{K;E}(\lambda) = \left(1 + \frac{3}{4}\lambda\right)^2 \left(1 + \frac{1}{4}\lambda\right),$$

and hence $\operatorname{vol}(K \sim \lambda E) = f_{K;E}(-\lambda)$. Finally, it is clear that $(4/3)E$ is not a summand of K, which concludes the proof. □

3.2 Characterizing sausages and linearity at one point

In this section we provide several characterizations of sausages which rely on the linearity of the volume (cf. Proposition 3.1.1) and some additional assumption on common/equal-volume projection or maximal volume section through parallel hyperplanes to a given one.

We will prove that the sole assumption of linearity at one point, together with the equal 'size' of a projection or a section, in the already mentioned sense, allows to characterize sausages.

In general, linearity of the volume at some point $\lambda_0 \in (0, 1)$ does not imply linearity of the volume. Indeed, if we take $\mathrm{W}_0 = 5, \mathrm{W}_1 = 4, \mathrm{W}_2 = 2$ and $\mathrm{W}_3 = 1$, these numbers satisfy inequalities (1.20) and hence there exist convex bodies $K, E \in \mathcal{K}^3$ such that $\mathrm{W}_i(K; E) = \mathrm{W}_i$, which yields $f_{K;E}(z) = 5 + 12z + 6z^2 + z^3$. So

$$f_{K;E}(-1) = -2, \quad f'_{K;E}(-1) = 3, \quad f''_{K;E}(-1) = 6, \quad f'''_{K;E}(-1) = 6,$$

and thus, by Lemma 3.1.1, $\operatorname{vol}\bigl(\lambda K + (1 - \lambda)E\bigr) = 1 + 3\lambda + 3\lambda^2 - 2\lambda^3$. Therefore, the volume of K_λ is not linear but satisfies

$$\operatorname{vol}\left(\frac{K + E}{2}\right) = 3 = \frac{1}{2}\mathrm{W}_0 + \frac{1}{2}\mathrm{W}_3 = \frac{1}{2}\operatorname{vol}(K) + \frac{1}{2}\operatorname{vol}(E),$$

i.e., there is linearity at $\lambda_0 = 1/2$.

In order to prove the main theorems of this section we still need further results, some of which already provide characterizations of sausages. We will see that, under the assumptions of common/equal-volume projection or maximum volume section, linearity of the volume at some point $\lambda_0 \in (0, 1)$ implies linearity of the volume.

Theorem 3.2.1 ([50]). *Let $K, E \in \mathcal{K}^n$ be convex bodies such that there is a hyperplane $H \in \mathcal{L}^n_{n-1}$ with $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(E|H)$. Then*

$$\mathrm{vol}\bigl(\lambda K + (1-\lambda)E\bigr) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E) \quad \textit{for all } \lambda \in [0,1],$$

if and only if either K and E lie in parallel hyperplanes, or the pair K, E is a sausage.

Proof. On account of Remark 3.2, we may assume without loss of generality that $\mathrm{vol}(K) > \mathrm{vol}(E)$ and also that $\mathrm{vol}_{n-1}(K|H) > 0$ (otherwise we would have $\mathrm{vol}(K) = \mathrm{vol}(E) = 0$).

Because of the linearity of the volume and by means of (3.3), we have that $f^{(n-j)}_{K;E}(-1) = 0$ for all $j \geq 2$ and $f^{(n-1)}_{K;E}(-1)/(n-1)! = \mathrm{vol}(K) - \mathrm{vol}(E)$, and thus

$$\begin{aligned} f_{K;E}(z) &= \mathrm{vol}(E)(z+1)^n + \bigl(\mathrm{vol}(K) - \mathrm{vol}(E)\bigr)(z+1)^{n-1} \\ &= \mathrm{vol}(E)z^n + \sum_{i=0}^{n-1} \left[\binom{n}{i}\mathrm{vol}(E) + \binom{n-1}{i}\bigl(\mathrm{vol}(K) - \mathrm{vol}(E)\bigr)\right] z^i. \end{aligned}$$

We define $l = \bigl(\mathrm{vol}(K) - \mathrm{vol}(E)\bigr)/\mathrm{vol}_{n-1}(K|H) > 0$ and $L = l\,[0,u]$, where $u \in \mathbb{S}^{n-1}$ is a normal vector of H, so having that

$$\mathrm{vol}(K) = \mathrm{vol}(E) + l\mathrm{vol}_{n-1}(K|H),$$

and thus, using (1.5) and (1.2), we get

$$\mathrm{vol}(L+E) = n\mathrm{W}_{n-1}(L;E) + \mathrm{vol}(E) = l\mathrm{vol}_{n-1}(E|H) + \mathrm{vol}(E) = l\mathrm{vol}_{n-1}(K|H) + \mathrm{vol}(E) = \mathrm{vol}(K).$$

Therefore we have

$$\begin{aligned} \mathrm{V}(K,\dots,K,L+E)^n &= \bigl(\mathrm{V}(K,\dots,K,L) + \mathrm{W}_1(K;E)\bigr)^n \\ &= \left(\frac{l\mathrm{vol}_{n-1}(K|H)}{n} + \frac{n\mathrm{vol}(E) + (n-1)l\mathrm{vol}_{n-1}(K|H)}{n}\right)^n \\ &= \mathrm{vol}(K)^n = \mathrm{vol}(K)^{n-1}\,\mathrm{vol}(L+E), \end{aligned}$$

and hence, by the equality case in Minkowski's first inequality (see Theorem 1.4.7) together with the common volume projection hypothesis, $K = L + E$ (up to translations).

The converse is immediately satisfied (cf. (3.4)). □

We notice that if $K = L + E$, with $L \in \mathcal{K}^n$, $\dim L \leq 1$, then $K|H = E|H$ for $H = L^\perp$. Besides, if K, E lie in parallel hyperplanes H_1 and H_2, then for any $H = u^\perp$, where u is a vector (embedded) in H_i, $i = 1, 2$, we have $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(E|H) = 0$. So the following result holds.

Theorem 3.2.2 ([50]). *Let $K, E \in \mathcal{K}^n$ be convex bodies. Then we have*

$$\mathrm{vol}\bigl(\lambda K + (1-\lambda)E\bigr) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E) \textit{ for all } \lambda \in [0,1], \textit{ and}$$
$$\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(E|H) \quad \textit{for some hyperplane } H \in \mathcal{L}^n_{n-1},$$

if and only if either K and E lie in parallel hyperplanes, or the pair K, E is a sausage.

In the following (Theorem 3.2.3), we will show that replacing a common volume projection by a common maximal volume section through parallel hyperplanes to a given one, we may obtain the same characterization. The Schwarz symmetrization will become an essential tool in order to exchange these above-mentioned assumptions. To this end, first we will prove a sufficient condition, relying on the Schwarz symmetrization, for the pair K, E to be a sausage.

Lemma 3.2.1. *Let $K, E \in \mathcal{K}^n$ be convex bodies, $\operatorname{int} K \neq \emptyset$, and let $H \in \mathcal{L}^n_{n-1}$ be a hyperplane. If*

$$\sigma_{H^\perp}\big(\lambda_0 K + (1-\lambda_0)E\big) = \lambda_0 \sigma_{H^\perp}(K) + (1-\lambda_0)\sigma_{H^\perp}(E) \tag{3.7}$$

for some $\lambda_0 \in (0,1)$ and

$$\sigma_{H^\perp}(K) = L + \sigma_{H^\perp}(E), \text{ where } L \in \mathcal{K}^n \text{ with } \dim L \leq 1, \tag{3.8}$$

then K is a sausage with respect to E.

Proof. We may assume that the origin is an interior point of K. By an appropriate choice of the coordinate axes, we may suppose that $H = \big\{(x_1, \ldots, x_n)^\intercal \in \mathbb{R}^n : x_1 = 0\big\}$. By definition of the Schwarz symmetrization, $L \subset H^\perp$ and then $L = \big[(a, 0, \ldots, 0)^\intercal, (b, 0, \ldots, 0)^\intercal\big]$, for some $a \leq b$.

We will denote by $H_t = \big\{(x_1, \ldots, x_n)^\intercal \in \mathbb{R}^n : x_1 = t\big\}$ and $H_t^+ = \big\{(x_1, \ldots, x_n)^\intercal \in \mathbb{R}^n : x_1 \geq t\big\}$ (respectively, $H_t^- = \big\{(x_1, \ldots, x_n)^\intercal \in \mathbb{R}^n : x_1 \leq t\big\}$) and, for any convex body M, by $M_t = M \cap H_t$ and $M_t^+ = M \cap H_t^+$ (respectively $M_t^- = M \cap H_t^-$, see Figure 3.2).

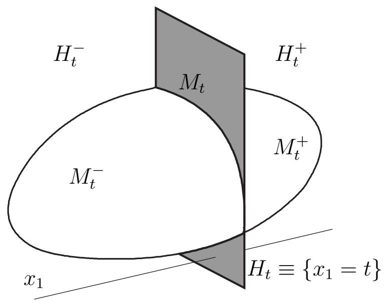

Figure 3.2: Dividing a convex body in pieces.

Without loss of generality, we may also assume that (one of) the maximum volume section(s) of E through hyperplanes parallel to H contains the origin. So, condition (3.8) implies that

$$\max_{x \in H^\perp} \operatorname{vol}_{n-1}\big(K \cap (x+H)\big) = \max_{x \in H^\perp} \operatorname{vol}_{n-1}\big(E \cap (x+H)\big) = m > 0$$

(since 0 is an interior point of K) and also that $\operatorname{vol}_{n-1}(K_t) = m$ for all $t \in [a, b]$.

Moreover, from the inclusion $K_{\lambda a + (1-\lambda)b} \supset \lambda K_a + (1-\lambda)K_b$, for $\lambda \in [0,1]$, and using Brunn-Minkowski's inequality in $\mathbb{R}^{n-1}$ we get

$$\begin{aligned} m &= \operatorname{vol}_{n-1}\left(K_{\lambda a + (1-\lambda)b}\right) \geq \operatorname{vol}_{n-1}\big(\lambda K_a + (1-\lambda)K_b\big) \\ &\geq \Big(\lambda \operatorname{vol}_{n-1}(K_a)^{1/(n-1)} + (1-\lambda)\operatorname{vol}_{n-1}(K_b)^{1/(n-1)}\Big)^{n-1} = m, \end{aligned}$$

and hence the equality case in Brunn-Minkowski's inequality allows to conclude that

$$K_{\lambda a+(1-\lambda)b} = K_a \quad \text{for all } \lambda \in [0,1] \tag{3.9}$$

(up to translations). Finally, we have to study what happens on the 'leftmost and rightmost parts' of K. To this aim, using Lemma 1.3.1 *i*) and the inclusion

$$\lambda_0 K_b^+ + (1-\lambda_0)E_0^+ \subset \big(\lambda_0 K + (1-\lambda_0)E\big)_{\lambda_0 b}^+,$$

we obtain, on the one hand,

$$\begin{aligned}
\mathrm{vol}\big(\lambda_0 K_b^+ + (1-\lambda_0)E_0^+\big) &= \mathrm{vol}\big(\sigma_{H^\perp}(\lambda_0 K_b^+ + (1-\lambda_0)E_0^+)\big)\\
&\le \mathrm{vol}\Big(\sigma_{H^\perp}\big((\lambda_0 K + (1-\lambda_0)E)_{\lambda_0 b}^+\big)\Big)\\
&= \mathrm{vol}\Big(\sigma_{H^\perp}\big(\lambda_0 K + (1-\lambda_0)E\big)_{\lambda_0 b}^+\Big)\\
&= \mathrm{vol}\Big(\big(\lambda_0\sigma_{H^\perp}(K) + (1-\lambda_0)\sigma_{H^\perp}(E)\big)_{\lambda_0 b}^+\Big)\\
&= \mathrm{vol}\Big(\big(\lambda_0 L + \sigma_{H^\perp}(E)\big)_{\lambda_0 b}^+\Big)\\
&= \mathrm{vol}\big(\sigma_{H^\perp}(E)_0^+\big) = \mathrm{vol}\big(E_0^+\big).
\end{aligned} \tag{3.10}$$

On the other hand, Brunn-Minkowski's inequality yields

$$\begin{aligned}
\mathrm{vol}\big(\lambda_0 K_b^+ + (1-\lambda_0)E_0^+\big) &\ge \Big(\lambda_0 \mathrm{vol}\big(K_b^+\big)^{1/n} + (1-\lambda_0)\mathrm{vol}\big(E_0^+\big)^{1/n}\Big)^n\\
&= \Big(\lambda_0 \mathrm{vol}\big(\sigma_{H^\perp}(K)_b^+\big)^{1/n} + (1-\lambda_0)\mathrm{vol}\big(E_0^+\big)^{1/n}\Big)^n\\
&= \Big(\lambda_0 \mathrm{vol}\big(\sigma_{H^\perp}(E)_0^+\big)^{1/n} + (1-\lambda_0)\mathrm{vol}\big(E_0^+\big)^{1/n}\Big)^n\\
&= \mathrm{vol}\big(E_0^+\big),
\end{aligned} \tag{3.11}$$

and hence, from (3.10) and (3.11) we have equality in Brunn-Minkowski's inequality for K_b^+ and E_0^+. Therefore, there are two possibilities depending on the dimension of E_0^+ and K_b^+:

i) If $\mathrm{vol}(E_0^+) = \mathrm{vol}(K_b^+) = 0$, then

$$\begin{aligned}
\mathrm{vol}_{n-1}\big(\lambda_0 K_b + (1-\lambda_0)E_0\big) &= \mathrm{vol}_{n-1}\big((\lambda_0 K + (1-\lambda_0)E)_{\lambda_0 b}\big)\\
&= \mathrm{vol}_{n-1}\big(\sigma_{H^\perp}(\lambda_0 K + (1-\lambda_0)E)_{\lambda_0 b}\big)\\
&= \mathrm{vol}_{n-1}\big((\lambda_0 L + \sigma_{H^\perp}(E))_{\lambda_0 b}\big)\\
&= \mathrm{vol}_{n-1}(E_0) = m > 0,
\end{aligned}$$

and thus (again by the equality case in Brunn-Minkowski's inequality) $K_b = y_0 + E_0$, for some $y_0 \in \mathbb{R}^n$. Hence $K_b^+ = K_b = y_0 + E_0 = y_0 + E_0^+$.

ii) If $\mathrm{vol}(E_0^+), \mathrm{vol}(K_b^+) > 0$ then, since they are homothetic with the same volume, $K_b^+ = y_0 + E_0^+$ for some $y_0 \in \mathbb{R}^n$.

In any case we have that $K_b^+ = y_0 + E_0^+$ for some $y_0 \in \mathbb{R}^n$ and, arguing in the same way as before, we may assert that $K_a^- = x_0 + E_0^-$ for some $x_0 \in \mathbb{R}^n$. These facts together with (3.9) (and by convexity) imply that $K = [x_0, y_0] + E$, i.e., K is a sausage with respect to E. $\square$

Remark 3.5. *We might wonder whether (only) one of the conditions* (3.7), (3.8) *is enough in order to characterize sausages. The answer is negative in both cases.*

i) *For* (3.7), *we consider $E = B_n$ and $K = L + B_{n-1}$ a cylinder, where $L \in \mathcal{K}^n$ with* $\dim L = 1$ *and $L \perp$* $\operatorname{aff} B_{n-1}$. *Since both sets (and their convex combination) are rotationally symmetric about the axis determined by L, it is clear that condition* (3.7) *holds for all $\lambda \in [0,1]$, but K is not a sausage with respect to E.*

ii) *For* (3.8), *we may consider $E = C_n$ and $K = L + C_{n-1}$ a parallelepiped, where L is a segment of appropriate length which is neither orthogonal nor parallel to* $\operatorname{aff} C_{n-1}$. *These bodies satisfy* (3.8) *for $H = \operatorname{aff} C_{n-1}$ but K is not a sausage with respect to E.*

We notice that in the above two examples it is also fulfilled that

$$\max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(K \cap (x + H)\big) = \max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(E \cap (x + H)\big), \tag{3.12}$$

for some hyperplane H. So even under this additional assumption, none of conditions (3.7), (3.8) is enough to determine sausages. However, as we shall see in the following result, (3.12) together with linearity of the volume allows to characterize sausages.

Theorem 3.2.3 ([50]). *Let $K, E \in \mathcal{K}^n$ be such that there exists a hyperplane $H \in \mathcal{L}^n_{n-1}$ with*

$$\max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(K \cap (x + H)\big) = \max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(E \cap (x + H)\big).$$

Then we have

$$\mathrm{vol}\big(\lambda K + (1-\lambda)E\big) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E) \quad \text{for all } \lambda \in [0,1],$$

if and only if either K and E lie in parallel hyperplanes, or the pair K, E is a sausage.

Proof. On account of Remark 3.2, we may assume without loss of generality that $\mathrm{vol}(K) > \mathrm{vol}(E)$. Denoting by $\nu = \max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(K \cap (x + H)\big)$, we get that the orthogonal projections onto H of the Schwarz symmetrals of K and E with respect to $H^\perp$, namely, $\sigma_{H^\perp}(K)$, $\sigma_{H^\perp}(E)$, are equal; more precisely,

$$\sigma_{H^\perp}(K)|H = \left(\frac{\nu}{\kappa_{n-1}}\right)^{1/(n-1)} B_{n-1} = \sigma_{H^\perp}(E)|H.$$

Thus, we can apply Theorem 2.1.2 to the convex bodies $\sigma_{H^\perp}(K)$, $\sigma_{H^\perp}(E)$ which, together with Lemma 1.3.1, $i), ii)$ yields

$$\begin{aligned}\mathrm{vol}\big(\lambda K + (1-\lambda)E\big) &= \mathrm{vol}\big(\sigma_{H^\perp}(\lambda K + (1-\lambda)E)\big) \geq \mathrm{vol}\big(\lambda\sigma_{H^\perp}(K) + (1-\lambda)\sigma_{H^\perp}(E)\big) \\ &\geq \lambda \mathrm{vol}\big(\sigma_{H^\perp}(K)\big) + (1-\lambda)\mathrm{vol}\big(\sigma_{H^\perp}(E)\big) = \lambda \mathrm{vol}(K) + (1-\lambda)\mathrm{vol}(E).\end{aligned}$$

Thus, the linearity of the volume for the bodies K, E implies, on the one hand, that

$$\sigma_{H^\perp}\big(\lambda K + (1-\lambda)E\big) = \lambda\sigma_{H^\perp}(K) + (1-\lambda)\sigma_{H^\perp}(E).$$

On the other hand, the linearity for the volume of the bodies $\sigma_{H^\perp}(K), \sigma_{H^\perp}(E)$ is also obtained, which, by Theorem 3.2.1, yields

$$\sigma_{H^\perp}(K) = L + \sigma_{H^\perp}(E), \text{ where } L \in \mathcal{K}^n \text{ with } \dim L \le 1.$$

Now, the result follows directly from Lemma 3.2.1. □

In order to reduce the assumption on the linearity of the volume for the range $[0,1]$ to a single point in $(0,1)$ we need first the following result, where not just equal volume projections are needed, but common projections of K and E.

Lemma 3.2.2 ([50]). *Let $K, E \in \mathcal{K}^n$ be convex bodies such that there exists a hyperplane $H \in \mathcal{L}^n_{n-1}$ with $K|H = E|H$. Then we have*

$$\mathrm{vol}\big(\lambda_0 K + (1-\lambda_0)E\big) = \lambda_0 \mathrm{vol}(K) + (1-\lambda_0)\mathrm{vol}(E), \quad \textit{for some } \lambda_0 \in (0,1),$$

if and only if either K and E lie in parallel hyperplanes, or the pair K, E is a sausage.

Proof. Since $K|H = E|H$, the function $\mathrm{V}_{K;E}(\lambda)$ is concave (see the proof of Theorem 2.1.3) which, together with linearity at λ_0, implies that $\mathrm{V}_{K;E}(\lambda)$ is an affine function on $[0,1]$ (see Remark 1.1). Now, the result follows from Theorem 3.2.1. □

Theorem 3.2.4 ([50]). *Let $K, E \in \mathcal{K}^n$ be convex bodies such that there exists a hyperplane $H \in \mathcal{L}^n_{n-1}$ with $\mathrm{vol}_{n-1}(K|H) = \mathrm{vol}_{n-1}(E|H)$. Then we have*

$$\mathrm{vol}\big(\lambda_0 K + (1-\lambda_0)E\big) = \lambda_0 \mathrm{vol}(K) + (1-\lambda_0)\mathrm{vol}(E) \quad \textit{for some } \lambda_0 \in (0,1),$$

if and only if either K and E lie in parallel hyperplanes, or the pair K, E is a sausage.

Proof. Without loss of generality (see Remark 3.2), we may assume that $\mathrm{vol}(K) > \mathrm{vol}(E)$. Using Lemma 1.3.1 *iii)*, we have that

$$\sigma_{H^\perp}\big(\sigma_H(K)\big)|H = \sigma_{H^\perp}\big(\sigma_H(E)\big)|H.$$

So, we can apply Theorem 2.1.2 to the convex bodies $\sigma_{H^\perp}\big(\sigma_H(K)\big)$, $\sigma_{H^\perp}\big(\sigma_H(E)\big)$ which, together with Lemma 1.3.1 *i)*, *ii)*, yields

$$\begin{aligned}
\mathrm{vol}\big(\lambda_0 K + (1-\lambda_0)E\big) &= \mathrm{vol}\Big(\sigma_{H^\perp}\Big(\sigma_H\big(\lambda_0 K + (1-\lambda_0)E\big)\Big)\Big)\\
&\ge \mathrm{vol}\Big(\lambda_0\sigma_{H^\perp}\big(\sigma_H(K)\big) + (1-\lambda_0)\sigma_{H^\perp}\big(\sigma_H(E)\big)\Big)\\
&\ge \lambda_0\mathrm{vol}\Big(\sigma_{H^\perp}\big(\sigma_H(K)\big)\Big) + (1-\lambda_0)\mathrm{vol}\Big(\sigma_{H^\perp}\big(\sigma_H(E)\big)\Big)\\
&= \lambda_0\mathrm{vol}(K) + (1-\lambda_0)\mathrm{vol}(E).
\end{aligned} \tag{3.13}$$

Thus, the linearity of the volume at λ_0 for the bodies K, E is equivalent to the same property for $\sigma_{H^\perp}\big(\sigma_H(K)\big), \sigma_{H^\perp}\big(\sigma_H(E)\big)$ and hence, by Lemma 3.2.2, we obtain

$$\sigma_{H^\perp}\big(\sigma_H(K)\big) = L + \sigma_{H^\perp}\big(\sigma_H(E)\big), \text{ with } L \subset H^\perp, \quad \dim L = 1, \quad \text{and} \tag{3.14}$$

$$\sigma_{H^\perp}\big(\lambda_0\sigma_H(K) + (1-\lambda_0)\sigma_H(E)\big) = \lambda_0\sigma_{H^\perp}\big(\sigma_H(K)\big) + (1-\lambda_0)\sigma_{H^\perp}\big(\sigma_H(E)\big). \tag{3.15}$$

We observe that $\dim L \neq 0$ because of the condition $\mathrm{vol}(K) > \mathrm{vol}(E)$. Now, the above conditions (3.14), (3.15) yield, by Lemma 3.2.1,

$$\sigma_H(K) = L_1 + \sigma_H(E), \quad \dim L_1 = 1,$$

where, from the common/equal-volume projection hypothesis, $L_1 \perp H$. Therefore, we have

$$K|H = \sigma_H(K)|H = \big(L_1 + \sigma_H(E)\big)|H = \sigma_H(E)|H = E|H$$

(up to translations), and hence Lemma 3.2.2 allows to assert that $K = L_0 + E$, $\dim L_0 = 1$. □

Replacing the common/equal-volume projection by an equal maximal volume section through parallel hyperplanes to a given one, we obtain the same characterization.

Theorem 3.2.5 ([50]). *Let $K, E \in \mathcal{K}^n$ be such that there exists a hyperplane $H \in \mathcal{L}^n_{n-1}$ with*

$$\max_{x\in H^\perp} \mathrm{vol}_{n-1}\big(K \cap (x+H)\big) = \max_{x\in H^\perp} \mathrm{vol}_{n-1}\big(E \cap (x+H)\big).$$

Then we have

$$\mathrm{vol}\big(\lambda_0 K + (1-\lambda_0)E\big) = \lambda_0\mathrm{vol}(K) + (1-\lambda_0)\mathrm{vol}(E), \quad \textit{for some } \lambda_0 \in (0,1),$$

if and only if either K and E lie in parallel hyperplanes, or the pair K, E is a sausage.

Proof. On account of Remark 3.2, we may assume without loss of generality that $\mathrm{vol}(K) > \mathrm{vol}(E)$. Arguing like in (3.13) with the sets $\sigma_{H^\perp}(K), \sigma_{H^\perp}(E)$, and by Lemma 3.2.2, we get that $\sigma_{H^\perp}(K)$ is a sausage with respect to $\sigma_{H^\perp}(E)$ and that $\sigma_{H^\perp}\big(\lambda_0 K + (1-\lambda_0)E\big) = \lambda_0\sigma_{H^\perp}(K) + (1-\lambda_0)\sigma_{H^\perp}(E)$. Hence, by Lemma 3.2.1, we may conclude that K is also a sausage with respect to E. □

Remark 3.6. *We would like to point out that, after the elaboration of the original work [50], the recent article [10] was brought to our attention.*

At that point, we realized that, for the particular case in which K and E are both n-dimensional convex bodies, Theorems 3.2.4 and 3.2.5 follow respectively from Theorem 1.4 and Theorem 1.5 in [10]. Nevertheless, the general results, as stated in our work, cannot be obtained from the above mentioned paper [10]. Therefore in order to deal with the most general cases, a different strategy seems to be needed, as shown here. Moreover, we would like to underline that, as it has been shown along this work, we have come to these conclusions from a totally different approach and in any case, we provide alternative proofs of them.

To conclude this section, we show that if we assume linearity at some point $\lambda_0 \in (0,1)$ for all quermassintegrals, then all of them are really linear functionals.

Proposition 3.2.1 ([50]). *If there exists* $\lambda_0 \in (0,1)$ *such that*

$$\mathrm{W}_i\bigl(\lambda_0 K + (1-\lambda_0)E; E\bigr) = \lambda_0 \mathrm{W}_i(K;E) + (1-\lambda_0)\mathrm{W}_i(E;E), \quad \textit{for all } i = 0, \dots, n-2,$$

then $\mathrm{W}_i\bigl(\lambda K + (1-\lambda)E; E\bigr)$ *is linear for all* $i = 0, \dots, n$.

Proof. We will prove the result by induction on $j = n - i$.

If $j = 2$, then it follows trivially from the fact that $\mathrm{W}_{n-2}\bigl(\lambda K + (1-\lambda)E; E\bigr)$ is a polynomial of degree at most two which coincides with $\lambda \mathrm{W}_{n-2}(K;E) + (1-\lambda)\mathrm{W}_{n-2}(E;E)$ at (at least) the points $0, \lambda_0, 1$, and hence they are really the same polynomial.

Now we assume $2 < j+1 \le n$ and that the result is true for j, i.e.,

$$\mathrm{W}_{n-j}\bigl(\lambda K + (1-\lambda)E; E\bigr) = \lambda \mathrm{W}_{n-j}(K;E) + (1-\lambda)\mathrm{W}_{n-j}(E;E)$$

for all $\lambda \in [0,1]$. Then, by Lemma 3.1.1, we have that

$$f_{K;E}^{(n-j)}(-1) = \cdots = f_{K;E}^{(n-2)}(-1) = 0$$

and so $\mathrm{W}_{n-j-1}\bigl(\lambda K + (1-\lambda)E; E\bigr) = a + b\lambda + c\lambda^{j+1}$. From the identities at $0, \lambda_0$ and 1, it follows

$$\begin{aligned} a + \lambda_0(b+c) &= \mathrm{W}_{n-j-1}(E;E) + \lambda_0\bigl(\mathrm{W}_{n-j-1}(K;E) - \mathrm{W}_{n-j-1}(E;E)\bigr) \\ &= \mathrm{W}_{n-j-1}\bigl(\lambda_0 K + (1-\lambda_0)E; E\bigr) = a + b\lambda_0 + c\lambda_0^{j+1}, \end{aligned}$$

so having that $c = 0$, and thus

$$\mathrm{W}_{n-j-1}\bigl(\lambda K + (1-\lambda)E; E\bigr) = a + b\lambda = \lambda \mathrm{W}_{n-j-1}(K;E) + (1-\lambda)\mathrm{W}_{n-j-1}(E;E),$$

which concludes the proof. □

3.3 Characterizing sausages via inequalities and roots of Steiner polynomials

We recall that the well-known Minkowski first inequality (see (1.25)) states that

$$\mathrm{V}\bigl(K[n-1], E\bigr)^n \ge \mathrm{vol}(K)^{n-1}\mathrm{vol}(E),$$

and equality holds, for $K, E \in \mathcal{K}_0^n$, if and only if K and E are homothetic.

In this section, first, we deal with the corresponding refinement of the above inequality, when working with additional projections/sections assumptions. To this aim, and for the sake of brevity, we will write

$$\mathrm{S}(K;E) = n\mathrm{W}_1(K;E) = n\mathrm{V}\bigl(K[n-1], E\bigr),$$

following the standard notation for the surface area. The main result is the following one:

Theorem 3.3.1 ([62]). *Let $K,E\in\mathcal{K}^n$ be convex bodies such that there exists a hyperplane $H\in\mathcal{L}^n_{n-1}$ with $\mathrm{vol}_{n-1}(K|H)=\mathrm{vol}_{n-1}(E|H)$. Then*

$$\mathrm{S}(K;E)\geq(n-1)\mathrm{vol}(K)+\mathrm{vol}(E), \tag{3.16}$$

and equality holds, for $K,E\in\mathcal{K}^n_0$, if and only if the pair K,E is a sausage.

We notice first that the above inequality is indeed stronger than (1.25) since, by the arithmetic-geometric mean inequality (1.30), we have $(n-1)\mathrm{vol}(K)+\mathrm{vol}(E)\geq n\mathrm{vol}(K)^{(n-1)/n}\mathrm{vol}(E)^{1/n}$.

We present here two proofs of (3.16). The first one, shorter and perhaps more elegant, has the disadvantage that it does not allow to characterize the equality case.

First proof. Using the (relative) Steiner formula (cf. (1.5)) it is easy to check that

$$\mathrm{S}(K;E)=\lim_{\lambda\to0^+}\frac{\mathrm{vol}\big((1-\lambda)K+\lambda E\big)-\mathrm{vol}\big((1-\lambda)K\big)}{\lambda}. \tag{3.17}$$

Thus, applying Theorem 2.1.2, we have

$$\begin{aligned}\mathrm{S}(K;E)&=\lim_{\lambda\to0^+}\frac{\mathrm{vol}\big((1-\lambda)K+\lambda E\big)-\mathrm{vol}\big((1-\lambda)K\big)}{\lambda}\\&\geq\lim_{\lambda\to0^+}\frac{(1-\lambda)\mathrm{vol}(K)+\lambda\mathrm{vol}(E)-(1-\lambda)^n\mathrm{vol}(K)}{\lambda}\\&=(n-1)\mathrm{vol}(K)+\mathrm{vol}(E).\end{aligned}$$

□

For the study of the equality case we will need a different proof. The first part of it follows the same steps to that of Theorem 7.2.1 in [52].

Before beginning with the proof, we point out the behavior of the 'relative surface area' $\mathrm{S}(K;E)$ with respect to the Schwarz symmetrization: using the well-known representation

$$\mathrm{S}(K;E)=\lim_{\lambda\to0^+}\frac{\mathrm{vol}(K+\lambda E)-\mathrm{vol}(K)}{\lambda}$$

(see e.g. (5.33) in [52], cf. (1.5) and (3.17)) and Lemma 1.3.1 *i), ii)*, it is immediate that

$$\begin{aligned}\mathrm{S}\big(\sigma_H(K);\sigma_H(E)\big)&=\lim_{\lambda\to0^+}\frac{\mathrm{vol}\big(\sigma_H(K)+\lambda\sigma_H(E)\big)-\mathrm{vol}\big(\sigma_H(K)\big)}{\lambda}\\&\leq\lim_{\lambda\to0^+}\frac{\mathrm{vol}\big(\sigma_H(K+\lambda E)\big)-\mathrm{vol}\big(\sigma_H(K)\big)}{\lambda}\\&=\lim_{\lambda\to0^+}\frac{\mathrm{vol}(K+\lambda E)-\mathrm{vol}(K)}{\lambda}=\mathrm{S}(K;E).\end{aligned} \tag{3.18}$$

Second proof. First, we will assume that $K|H=E|H$. In this case, the function $f:[0,1]\longrightarrow\mathbb{R}_{\geq0}$ defined by $f(t)=\mathrm{vol}\big((1-t)K+tE\big)-(1-t)\mathrm{vol}(K)-t\mathrm{vol}(E)$ is concave by Theorem 2.1.1 and it satisfies $f(0)=f(1)=0$. Hence, its derivative at 0,

$$f'(0)=\mathrm{S}(K;E)-\big((n-1)\mathrm{vol}(K)+\mathrm{vol}(E)\big),$$

fulfills $f'(0)\geq 0$, which shows (3.16) in this case, and $f'(0)=0$ if and only if f is identically 0. The latter implies equality in (2.1), on the whole interval $[0,1]$, and thus Theorem 3.2.1 ensures that, for $K,E\in\mathcal{K}^n_0$, the pair K,E is a sausage.

Now we deal with the general case $\mathrm{vol}_{n-1}(K|H)=\mathrm{vol}_{n-1}(E|H)$. Applying Schwarz symmetrizations with respect to H and $H^{\perp}$ respectively, and since $\sigma_{H^\perp}\big(\sigma_H(K)\big)|H=\sigma_{H^\perp}\big(\sigma_H(E)\big)|H$, we get, using (3.18), that

$$\begin{aligned}\mathrm{S}(K;E)&\geq \mathrm{S}\big(\sigma_H(K);\sigma_H(E)\big)\geq \mathrm{S}\Big(\sigma_{H^\perp}\big(\sigma_H(K)\big);\sigma_{H^\perp}\big(\sigma_H(E)\big)\Big)\\&\geq (n-1)\mathrm{vol}\Big(\sigma_{H^\perp}\big(\sigma_H(K)\big)\Big)+\mathrm{vol}\Big(\sigma_{H^\perp}\big(\sigma_H(E)\big)\Big)\\&=(n-1)\mathrm{vol}(K)+\mathrm{vol}(E),\end{aligned}$$

where equality holds, for $K,E\in\mathcal{K}^n_0$, if and only if

$$\mathrm{S}(K;E)=\mathrm{S}\big(\sigma_H(K);\sigma_H(E)\big)=\mathrm{S}\Big(\sigma_{H^\perp}\big(\sigma_H(K)\big);\sigma_{H^\perp}\big(\sigma_H(E)\big)\Big) \tag{3.19}$$

and (by the previous case) the pair $\sigma_{H^\perp}\big(\sigma_H(K)\big),\sigma_{H^\perp}\big(\sigma_H(E)\big)$ is a sausage.

We notice that since (3.16) is not symmetric on the bodies K,E, in order to deal with the equality case, we should distinguish two cases:

Case 1: $\sigma_{H^\perp}\big(\sigma_H(K)\big)$ is a sausage with respect to $\sigma_{H^\perp}\big(\sigma_H(E)\big)$, i.e., $\sigma_{H^\perp}\big(\sigma_H(K)\big)=L+\sigma_{H^\perp}\big(\sigma_H(E)\big)$, $L\in\mathcal{K}^n$ with $\dim L\leq 1$.

By an appropriate choice of the coordinate axes, we may suppose that the hyperplane H is given by $H=\big\{(x_1,\dots,x_n)^\intercal\in\mathbb{R}^n:\,x_1=0\big\}$. By definition of the Schwarz symmetrization, $L\subset H^\perp$ and $L=\big[(a,0,\dots,0)^\intercal,(b,0,\dots,0)^\intercal\big]$ for some $a\leq b$. For short we write $K'=\sigma_H(K)$ and $E'=\sigma_H(E)$. On the one hand, since

$$\sigma_{H^\perp}(K')=L+\sigma_{H^\perp}(E'), \tag{3.20}$$

following the same notation as in Lemma 3.2.1 and by the equality case in Brunn-Minkowski's inequality (1.22) we may conclude that, up to translations, $(K')_t=(K')_a$ for all $t\in[a,b]$ (cf. (3.9)). In particular, there exists $x\in\mathbb{R}^n$ so that $(K')_b=x+(K')_a$. Hence, the convexity of K' allows to assert that $K'=\widetilde{L}+\widetilde{E}$ for some $\widetilde{L},\widetilde{E}\in\mathcal{K}^n$ with $\dim\widetilde{L}\leq 1$, $\dim\widetilde{E}=n$, and moreover,

$$\sigma_{H^\perp}(\widetilde{E})=\sigma_{H^\perp}(E') \tag{3.21}$$

because $\sigma_{H^\perp}(K')=L+\sigma_{H^\perp}(E')$.

On the other hand, since $K|H=K'|H=\widetilde{L}|H+\widetilde{E}|H$ and

$$\begin{aligned}\mathrm{vol}_{n-1}(K|H)&=\mathrm{vol}_{n-1}(E|H)=\mathrm{vol}_{n-1}(E'|H)\\&=\mathrm{vol}_{n-1}\big(\sigma_{H^\perp}(E')|H\big)=\mathrm{vol}_{n-1}\big(\sigma_{H^\perp}(\widetilde{E})|H\big)=\mathrm{vol}_{n-1}\big(\widetilde{E}|H\big)\end{aligned} \tag{3.22}$$

by (3.21), we may assure that $\widetilde{L}|H$ is a point, i.e., $\widetilde{L}\perp H$ (if $\widetilde{L}$ is not a point), and furthermore $\mathrm{vol}_1(\widetilde{L})=\mathrm{vol}_1(L)$.

Hence, using (3.20), (1.7) and (1.2), we have, on the one hand, that

$$\begin{aligned} \mathrm{S}\bigl(\sigma_{H^\perp}(K');\sigma_{H^\perp}(E')\bigr) &= \mathrm{S}\bigl(L+\sigma_{H^\perp}(E');\sigma_{H^\perp}(E')\bigr) = n\mathrm{W}_1\bigl(L+\sigma_{H^\perp}(E');\sigma_{H^\perp}(E')\bigr)\\ &= n\Bigl[\mathrm{vol}\bigl(\sigma_{H^\perp}(E')\bigr)+(n-1)\mathrm{W}_{n-1}\bigl(L;\sigma_{H^\perp}(E')\bigr)\Bigr]\\ &= n\mathrm{vol}(E')+(n-1)\mathrm{vol}_1(L)\mathrm{vol}_{n-1}\bigl(\sigma_{H^\perp}(E')|H\bigr)\\ &= n\mathrm{vol}(E')+(n-1)\mathrm{vol}_1(L)\mathrm{vol}_{n-1}(E'|H). \end{aligned} \tag{3.23}$$

On the other hand, by (1.3) and (1.2),

$$\begin{aligned} \mathrm{S}(K';E') &= \mathrm{S}\bigl(\widetilde{L}+\widetilde{E};E'\bigr) = n\mathrm{W}_1\bigl(\widetilde{L}+\widetilde{E};E'\bigr) = n\mathrm{V}\left(\bigl(\widetilde{L}+\widetilde{E}\bigr)[n-1],E'\right)\\ &= n\left[\mathrm{V}\bigl(\widetilde{E}[n-1],E'\bigr)+(n-1)\mathrm{V}\bigl(\widetilde{L},\widetilde{E}[n-2],E'\bigr)\right]\\ &= n\mathrm{V}\bigl(\widetilde{E}[n-1],E'\bigr)+(n-1)\,\mathrm{vol}_1\bigl(\widetilde{L}\bigr)\vartheta^{(n-1)}\left(\widetilde{E}|H[n-2],E'|H\right), \end{aligned}$$

and then, applying Minkowski's first inequality (1.25) in the above two mixed volumes, and since $\mathrm{vol}\bigl(\widetilde{E}\bigr)=\mathrm{vol}(E')$ (cf. (3.21)) and $\mathrm{vol}_{n-1}\bigl(\widetilde{E}|H\bigr)=\mathrm{vol}_{n-1}(E'|H)$ (cf. (3.22)), we obtain

$$\begin{aligned} \mathrm{S}(K';E') &= n\mathrm{V}\bigl(\widetilde{E}[n-1],E'\bigr)+(n-1)\,\mathrm{vol}_1\bigl(\widetilde{L}\bigr)\vartheta^{(n-1)}\left(\widetilde{E}|H[n-2],E'|H\right)\\ &\geq n\mathrm{vol}\bigl(\widetilde{E}\bigr)^{(n-1)/n}\mathrm{vol}(E')^{1/n}+(n-1)\mathrm{vol}_1\bigl(\widetilde{L}\bigr)\mathrm{vol}_{n-1}\bigl(\widetilde{E}|H\bigr)^{(n-2)/(n-1)}\mathrm{vol}_{n-1}(E'|H)^{1/(n-1)}\\ &= n\mathrm{vol}(E')+(n-1)\mathrm{vol}_1(L)\mathrm{vol}_{n-1}(E'|H). \end{aligned} \tag{3.24}$$

Finally, by (3.19), we can put together (3.23) and (3.24) to get that, necessarily, we have equality in Minkowski's first inequality (1.25), which implies that $\widetilde{E}=E'$ (since $\mathrm{vol}(\widetilde{E})=\mathrm{vol}(E')$), up to translations, and hence

$$\sigma_H(K)=K'=\widetilde{L}+E'=\widetilde{L}+\sigma_H(E),\quad \dim\widetilde{L}\leq 1,\quad \widetilde{L}\perp H.$$

Therefore, (up to translations) we have

$$K|H=\sigma_H(K)|H=\bigl(\widetilde{L}+\sigma_H(E)\bigr)|H=\sigma_H(E)|H=E|H$$

and thus, from the equality case for bodies which have a common projection onto a hyperplane, we may conclude that $K=L_0+E$ with $L_0\in\mathcal{K}^n$, $\dim L_0\leq 1$.

Case 2: $\sigma_{H^\perp}\bigl(\sigma_H(E)\bigr)$ is a sausage with respect to $\sigma_{H^\perp}\bigl(\sigma_H(K)\bigr)$, i.e., $\sigma_{H^\perp}\bigl(\sigma_H(E)\bigr)=L+\sigma_{H^\perp}\bigl(\sigma_H(K)\bigr)$, $L\in\mathcal{K}^n$ with $\dim L\leq 1$.

Arguing in the same way as in **Case 1** (and following the same notation, but interchanging K and E) we get $E'=\widetilde{L}+\widetilde{K}$ for some $\widetilde{L},\widetilde{K}\in\mathcal{K}^n$ with $\dim\widetilde{L}\leq 1$, $\dim\widetilde{K}=n$, and $\sigma_{H^\perp}(\widetilde{K})=\sigma_{H^\perp}(K')$. And it is obtained, moreover, that $\widetilde{L}\perp H$ and $\mathrm{vol}_1(\widetilde{L})=\mathrm{vol}_1(L)$. Thus, all together, and using again (1.2), (1.3), (1.25) and (3.19), we have

$$\begin{aligned} n\mathrm{vol}(K')+\mathrm{vol}_{n-1}(K'|H)\mathrm{vol}_1(L) &= \mathrm{S}\bigl(\sigma_{H^\perp}(K');L+\sigma_{H^\perp}(K')\bigr)=\mathrm{S}\bigl(\sigma_{H^\perp}(K');\sigma_{H^\perp}(E')\bigr)\\ &= \mathrm{S}(K';E')=\mathrm{S}\bigl(K';\widetilde{L}+\widetilde{K}\bigr)\\ &= n\mathrm{V}\bigl(K'[n-1],\widetilde{K}\bigr)+\mathrm{vol}_{n-1}(K'|H)\mathrm{vol}_1\bigl(\widetilde{L}\bigr)\\ &\geq n\mathrm{vol}(K')+\mathrm{vol}_{n-1}(K'|H)\mathrm{vol}_1(L). \end{aligned}$$

Therefore, we necessarily have equality in Minkowski's first inequality (1.25) which implies that, up to translations, $\widetilde{K} = K'$ (since $\mathrm{vol}(\widetilde{K}) = \mathrm{vol}(K')$). Finally, similarly to **Case 1**, we can conclude that $E = L_0 + K$ with $L_0 \in \mathcal{K}^n$, $\dim L_0 \leq 1$. This finishes the proof. □

Remark 3.7. *We would like to point out that the inequality obtained in the above theorem, as well as its equality case, can be obtained from inequality (7.191) and Theorem 7.7.3 in [52] for the case of* non-degenerated convex bodies having a common projection *onto a hyperplane.*

As in the case of the Brunn-Minkowski inequality, the same refinement can be deduced when exchanging common volume projections by common maximal volume sections through parallel hyperplanes to a given one. More precisely, we have the following result.

Theorem 3.3.2 ([62]). *Let $K, E \in \mathcal{K}^n$ be such that there exists a hyperplane $H \in \mathcal{L}^n_{n-1}$ with*

$$\max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(K \cap (x + H)\big) = \max_{x \in H^\perp} \mathrm{vol}_{n-1}\big(E \cap (x + H)\big). \tag{3.25}$$

Then

$$\mathrm{S}(K; E) \geq (n-1)\mathrm{vol}(K) + \mathrm{vol}(E),$$

and equality holds, for $K, E \in \mathcal{K}^n_0$, if and only if the pair K, E is a sausage.

For the proof of this theorem we will need the following well-known property relating volumes and projections (see e.g. page 106 in [21]): let $M, L \in \mathcal{K}^n$ with $\dim M = n-1$ and $\dim L = 1$. Then

$$\mathrm{vol}(M|L^\perp)\mathrm{vol}_1(L) = \mathrm{vol}_{n-1}(M)\mathrm{vol}_1\big(L|(\mathrm{aff}\, M)^\perp\big). \tag{3.26}$$

Proof of Theorem 3.3.2. Because of condition (3.25), the Schwarz symmetrals of K and E with respect to $H^\perp$ fulfills $\sigma_{H^\perp}(K)|H = \sigma_{H^\perp}(E)|H$, and thus, applying Theorem 3.3.1, we obtain

$$\mathrm{S}(K; E) \geq \mathrm{S}\big(\sigma_{H^\perp}(K); \sigma_{H^\perp}(E)\big) \geq (n-1)\mathrm{vol}\big(\sigma_{H^\perp}(K)\big) + \mathrm{vol}\big(\sigma_{H^\perp}(E)\big) = (n-1)\mathrm{vol}(K) + \mathrm{vol}(E),$$

and equality holds, for $K, E \in \mathcal{K}^n_0$, if and only if the pair $\sigma_{H^\perp}(K), \sigma_{H^\perp}(E)$ is a sausage and

$$\mathrm{S}(K; E) = \mathrm{S}\big(\sigma_{H^\perp}(K); \sigma_{H^\perp}(E)\big). \tag{3.27}$$

Now, the proof concludes in the same way as the one of Theorem 3.3.1, distinguishing two cases. We consider **Case 1**; the second case is analogous. Thus, we assume that $\sigma_{H^\perp}(K)$ is a sausage with respect to $\sigma_{H^\perp}(E)$, i.e., $\sigma_{H^\perp}(K) = L + \sigma_{H^\perp}(E)$ with $\dim L \leq 1$ (and $L \subset H^\perp$). Again, arguing as in the proof of Theorem 3.3.1, we get that $K_t = K_a$ (up to translations) for all $t \in [a, b]$. In particular, there exists $x \in \mathbb{R}^n$ so that $K_b = x + K_a$, and hence the convexity of K allows to assert that $K = \widetilde{L} + \widetilde{E}$ with $\dim \widetilde{L} \leq 1$, $\dim \widetilde{E} = n$, and

$$\sigma_{H^\perp}\big(\widetilde{E}\big) = \sigma_{H^\perp}(E). \tag{3.28}$$

We notice that, since there is no (common) projection assumption, we cannot assure neither $\widetilde{L} \perp H$ nor $\mathrm{vol}_1\big(\widetilde{L}\big) = \mathrm{vol}_1(L)$.

First we see that we can assume $\dim\widetilde{L}=1$. Indeed, if $\dim\widetilde{L}=0$ then $K=\widetilde{E}$ up to translations, which yields, by (3.28), $\sigma_{H^\perp}(K)=\sigma_{H^\perp}\bigl(\widetilde{E}\bigr)=\sigma_{H^\perp}(E)$. Thus, in particular, $\mathrm{vol}(K)=\mathrm{vol}(E)$, and using also (3.27) and Minkowski's first inequality (1.25) we get

$$\begin{aligned} n\mathrm{vol}(K) &= n\mathrm{vol}\bigl(\sigma_{H^\perp}(K)\bigr)=\mathrm{S}\bigl(\sigma_{H^\perp}(K),\sigma_{H^\perp}(K)\bigr)=\mathrm{S}\bigl(\sigma_{H^\perp}(K),\sigma_{H^\perp}(E)\bigr)=\mathrm{S}(K;E)\\ &= n\mathrm{V}\bigl(K[n-1],E\bigr)\geq n\mathrm{vol}(K). \end{aligned}$$

Therefore, we necessarily have equality in Minkowski's first inequality, which implies that, up to translations, $K=E$ (since $\mathrm{vol}(K)=\mathrm{vol}(E)$). It shows the result.

So, we suppose $\dim\widetilde{L}=1$. On the one hand we notice that, since $K=\widetilde{L}+\widetilde{E}$ is a sausage and $\sigma_{H^\perp}(K)=L+\sigma_{H^\perp}(E)=L+\sigma_{H^\perp}\bigl(\widetilde{E}\bigr)$ (cf. (3.28)), then $\sigma_{H^\perp}\bigl(\widetilde{L}\bigr)=L$, i.e.,

$$\widetilde{L}|H^\perp=L. \tag{3.29}$$

On the other hand, denoting by $\widetilde{E}_M$ (respectively, E_M) the 'maximal volume section' of $\widetilde{E}$ (respectively, E) with respect to the hyperplane H, i.e., $\mathrm{vol}_{n-1}\bigl(\widetilde{E}_M\bigr)=\max_{x\in H^\perp}\mathrm{vol}_{n-1}\bigl(\widetilde{E}\cap(x+H)\bigr)$ (analogously for E), from (3.28) we obtain that

$$\sigma_{H^\perp}\bigl(\widetilde{E}_M\bigr)=\sigma_{H^\perp}(E_M), \tag{3.30}$$

and moreover, the 'sausage property' of K implies that $\widetilde{E}|\widetilde{L}^\perp=\widetilde{E}_M|\widetilde{L}^\perp$, which yields

$$\mathrm{vol}_{n-1}\bigl(\widetilde{E}|\widetilde{L}^\perp\bigr)=\mathrm{vol}_{n-1}\bigl(\widetilde{E}_M|\widetilde{L}^\perp\bigr). \tag{3.31}$$

Then, since $E_M\subset E$, $\mathrm{aff}\,E_M$ is parallel to H, and using (3.26) and (3.29), we get, on the one hand,

$$\begin{aligned} \mathrm{vol}_{n-1}\bigl(E|\widetilde{L}^\perp\bigr)\mathrm{vol}_1\bigl(\widetilde{L}\bigr) &\geq \mathrm{vol}_{n-1}\bigl(E_M|\widetilde{L}^\perp\bigr)\mathrm{vol}_1\bigl(\widetilde{L}\bigr)=\mathrm{vol}_{n-1}(E_M)\mathrm{vol}_1\left(\widetilde{L}\middle|(\mathrm{aff}\,E_M)^\perp\right)\\ &= \mathrm{vol}_{n-1}(E_M)\mathrm{vol}_1(L). \end{aligned} \tag{3.32}$$

On the other hand, by (3.31), (3.26) and (3.30) we have

$$\begin{aligned} \mathrm{vol}_{n-1}\bigl(\widetilde{E}|\widetilde{L}^\perp\bigr)\mathrm{vol}_1\bigl(\widetilde{L}\bigr) &= \mathrm{vol}_{n-1}\bigl(\widetilde{E}_M|\widetilde{L}^\perp\bigr)\mathrm{vol}_1\bigl(\widetilde{L}\bigr)=\mathrm{vol}_{n-1}\bigl(\widetilde{E}_M\bigr)\mathrm{vol}_1(L)\\ &= \mathrm{vol}_{n-1}\bigl(\sigma_{H^\perp}(\widetilde{E}_M)\bigr)\mathrm{vol}_1(L)\\ &= \mathrm{vol}_{n-1}\bigl(\sigma_{H^\perp}(E_M)\bigr)\mathrm{vol}_1(L)=\mathrm{vol}_{n-1}(E_M)\mathrm{vol}_1(L), \end{aligned} \tag{3.33}$$

and thus, (3.32) and (3.33) together yield

$$\mathrm{vol}_{n-1}\bigl(\widetilde{E}|\widetilde{L}^\perp\bigr)\leq\mathrm{vol}_{n-1}\bigl(E|\widetilde{L}^\perp\bigr). \tag{3.34}$$

Now we have all the needed ingredients in order to argue like in the proof of Theorem 3.3.1: using (1.2), (1.3) and (1.25), and taking into account (3.27) and (3.34), we can deduce that

$$\begin{aligned} n\mathrm{vol}(E)+(n-1)\mathrm{vol}_{n-1}\bigl(\sigma_{H^\perp}(E)|H\bigr)\mathrm{vol}_1(L) &= \mathrm{S}\bigl(L+\sigma_{H^\perp}(E);\sigma_{H^\perp}(E)\bigr)\\ &= \mathrm{S}\bigl(\sigma_{H^\perp}(K);\sigma_{H^\perp}(E)\bigr)=\mathrm{S}(K;E)=\mathrm{S}\bigl(\widetilde{L}+\widetilde{E};E\bigr)\\ &= n\mathrm{V}\bigl(\widetilde{E}[n-1],E\bigr)+(n-1)\,\vartheta^{(n-1)}\bigl(\widetilde{E}|\widetilde{L}^\perp[n-2],E|\widetilde{L}^\perp\bigr)\mathrm{vol}_1\bigl(\widetilde{L}\bigr)\\ &\geq n\mathrm{vol}(E)+(n-1)\mathrm{vol}_{n-1}\bigl(\widetilde{E}|\widetilde{L}^\perp\bigr)\mathrm{vol}_1\bigl(\widetilde{L}\bigr), \end{aligned} \tag{3.35}$$

this is,

$$\mathrm{vol}_{n-1}\big(\sigma_{H^\perp}(E)|H\big)\mathrm{vol}_1(L) \geq \mathrm{vol}_{n-1}\big(\widetilde{E}|\widetilde{L}^\perp\big)\mathrm{vol}_1\big(\widetilde{L}\big).$$

Finally, using again (3.30), that aff E_M is parallel to H, (3.29), (3.26) and (3.31), we have that

$$\begin{aligned}
\mathrm{vol}_{n-1}\big(\sigma_{H^\perp}(E)|H\big)\mathrm{vol}_1(L) &= \max_{x\in H^\perp} \mathrm{vol}_{n-1}\big(E\cap(x+H)\big)\mathrm{vol}_1(L) = \mathrm{vol}_{n-1}(E_M)\mathrm{vol}_1(L)\\
&= \mathrm{vol}_{n-1}\big(\widetilde{E}_M\big)\mathrm{vol}_1(L) = \mathrm{vol}_{n-1}\big(\widetilde{E}_M\big)\mathrm{vol}_1\big(\widetilde{L}|(\mathrm{aff}\, E_M)^\perp\big)\\
&= \mathrm{vol}_{n-1}\big(\widetilde{E}_M|\widetilde{L}^\perp\big)\mathrm{vol}_1\big(\widetilde{L}\big)\\
&= \mathrm{vol}_{n-1}\big(\widetilde{E}|\widetilde{L}^\perp\big)\mathrm{vol}_1\big(\widetilde{L}\big),
\end{aligned}$$

i.e., we have equality in the previous inequality, and thus, by (3.35), we necessarily have equality in Minkowski's first inequality (1.25). So, since $\mathrm{vol}(\widetilde{E}) = \mathrm{vol}(E)$, it implies that $\widetilde{E} = E$ up to translations, this is, the pair K, E is a sausage, which concludes the proof of **Case 1**. As previously mentioned, the proof of **Case 2**, i.e., when $\sigma_{H^\perp}(E)$ is a sausage with respect to $\sigma_{H^\perp}(K)$, is analogous to the previous one, and we will not repeat it here (see also the proof of Theorem 3.3.1). □

By considering the special case where K or E a ball, (3.16) reduces to refinements of well-known and relevant geometric inequalities for convex bodies. We recall that

$$\mathrm{S}(K) = n\mathrm{W}_1(K) = n\mathrm{V}\big(K[n-1], B_n\big)$$

and

$$\mathrm{b}(K) = \frac{2}{\kappa_n}\mathrm{W}_{n-1}(K) = \frac{2}{\kappa_n}\mathrm{V}\big(B_n[n-1], K\big)$$

are respectively the surface area and the mean width of K.

Thus, on the one hand, (3.16) implies that, under a common (volume) projection hypothesis (resp. common maximal volume section hypothesis) of a convex body K with a ball rB_n, we have the following refinement of the isoperimetric inequality (1.26):

$$r\,\mathrm{S}(K) \geq (n-1)\mathrm{vol}(K) + \kappa_n r^n,$$

and equality holds, if and only if K is a sausage with respect to the ball rB_n. In the same way, we have the following linear version of Urysohn's inequality (see e.g. page 382 of [52]):

$$\frac{n\kappa_n}{2}\, r^{n-1}\mathrm{b}(K) \geq (n-1)r^n\kappa_n + \mathrm{vol}(K),$$

and equality holds, if and only if K is a sausage with respect to the ball rB_n.

Remark 3.8. *At this point we would like to observe that, following the first proof of Theorem 3.3.1, if we apply Proposition 2.2.1 instead of Theorem 2.1.2, we get that, for $k \geq 2$,*

$$\begin{aligned}
\mathrm{S}(K;E) &= \lim_{\lambda\to 0^+} \frac{\mathrm{vol}\big((1-\lambda)K+\lambda E\big) - \mathrm{vol}\big((1-\lambda)K\big)}{\lambda}\\
&\geq \lim_{\lambda\to 0^+} \frac{(1-\lambda)^k\mathrm{vol}(K) + \lambda^k\mathrm{vol}(E) - (1-\lambda)^n\mathrm{vol}(K)}{\lambda} = (n-k)\mathrm{vol}(K),
\end{aligned}$$

although this is not a sharp inequality.

3.3.1 Sausages via the roots of Steiner polynomials

Next, we deal with the problem of obtaining sausages via the roots of Steiner polynomials. As we have shown along this chapter (cf. Theorem 3.1.1), the fact of having $-\mathrm{r}(K)$ as an $(n-1)$-fold root of the Steiner polynomial (of the convex body $K \in \mathcal{K}^n$) is not enough to characterize sausages. Thus, we should assume an additional hypothesis in order to get such a characterization. Moreover, we would like to point out that, although we will work with the classical Steiner polynomial $f_{K;B_n}(z)$ (in order to avoid less elegant generalizations of the functionals $\mathrm{D}(\cdot), \mathrm{b}(\cdot)$), the results may be extended to the relative Steiner polynomial $f_{K;E}(z)$ with respect to any gauge body E. We will also assume, for the sake of brevity, that $\mathrm{r}(K) = 1$.

The following auxiliary lemma will be needed later on.

Lemma 3.3.1 ([62]). *Let $K \in \mathcal{K}^n$ be a convex body such that $-\mathrm{r}(K) = -1$ is an $(n-1)$-fold root of the Steiner polynomial $f_{K;B_n}(z)$ and let γ be the remaining (real) root. Then the real number*

$$l = -\frac{\kappa_n}{\kappa_{n-1}}(\gamma + 1) \tag{3.36}$$

is non-negative.

Proof. It is easy to see that

$$\begin{aligned} f_{K;B_n}(z) &= \kappa_n (z+1)^{n-1}(z-\gamma) = \kappa_n (z+1)^{n-1}\left(z + 1 + l\frac{\kappa_{n-1}}{\kappa_n}\right) \\ &= l\kappa_{n-1}(z+1)^{n-1} + \kappa_n (z+1)^n. \end{aligned} \tag{3.37}$$

Thus, equating coefficients, we get

$$\begin{aligned} n\mathrm{W}_{n-1}(K) &= l\kappa_{n-1} + n\kappa_n; \\ \binom{n}{2}\mathrm{W}_{n-2}(K) &= l(n-1)\kappa_{n-1} + \binom{n}{2}\kappa_n, \end{aligned} \tag{3.38}$$

and applying the inequality $\mathrm{W}_{n-1}(K) \le \mathrm{W}_{n-2}(K)$ (cf. (1.12)) we have

$$l\kappa_{n-1} + n\kappa_n = n\mathrm{W}_{n-1}(K) \le n\mathrm{W}_{n-2}(K) = 2l\kappa_{n-1} + n\kappa_n,$$

which implies $l \ge 0$. □

The following result allows to characterize sausages via the inradius $\mathrm{r}(K) = 1$ as an $(n-1)$-fold root of $f_{K;B_n}(z)$ together with a lower bound for the diameter of the so-called *kernel* $K_{-1} = K \sim B_n$ of K in terms of the remaining real root of $f_{K;B_n}(z)$.

Theorem 3.3.3 ([62]). *Let $K \in \mathcal{K}^n$ with $\mathrm{r}(K) = 1$. Then K is a sausage with respect to B_n if and only if -1 is an $(n-1)$-fold root of $f_{K;B_n}(z)$ and $\mathrm{D}(K_{-1}) \ge l$, where l is defined in (3.36).*

Proof. First, we suppose that $-\mathrm{r}(K) = -1$ is an $(n-1)$-fold root of $f_{K;B_n}(z)$ and $\mathrm{D}(K_{-1}) \ge l$. We distinguish two cases:

If $\dim K_{-1} = 0$, i.e., if K_{-1} is a point, then we get $0 = \mathrm{D}(K_{-1}) \geq l$, and Lemma 3.3.1 ensures that $l = 0$. So, $f_{K;B_n}(z) = \kappa_n(z+1)^n$, which implies that $K = B_n$ up to a translations (see Proposition 3.2 in [26]), i.e., K is a 'degenerated' sausage.

If $\dim K_{-1} \geq 1$, we suppose, by contradiction, that $\dim K_{-1} > 1$. Then there exist points $x, y \in K_{-1}$ such that the line segment $[x,y] \subsetneq K_{-1}$ and $\mathrm{vol}_1\bigl([x,y]\bigr) = \mathrm{D}(K_{-1}) \geq l$. Thus, we have $[x,y] + B_n \subsetneq K_{-1} + B_n \subset K$ and hence

$$\mathrm{vol}_1\bigl([x,y]\bigr)\kappa_{n-1} + \kappa_n = \mathrm{vol}\bigl([x,y] + B_n\bigr) < \mathrm{vol}(K) = f_{K;B_n}(0) = l\kappa_{n-1} + \kappa_n$$

(see (3.37)), i.e., $\mathrm{vol}_1\bigl([x,y]\bigr) < l$, contradicting our assumption $\mathrm{D}(K_{-1}) \geq l$.

So, $\dim K_{-1} = 1$ and hence

$$l\kappa_{n-1} + \kappa_n \leq \mathrm{D}(K_{-1})\kappa_{n-1} + \kappa_n = \mathrm{vol}_1(K_{-1})\kappa_{n-1} + \kappa_n = \mathrm{vol}(K_{-1} + B_n) \leq \mathrm{vol}(K) = l\kappa_{n-1} + \kappa_n.$$

Therefore, in particular, $\mathrm{vol}(K_{-1} + B_n) = \mathrm{vol}(K)$, which implies that $K = K_{-1} + B_n$ is a sausage (property (iv) of the volume, page 4).

Conversely, if K is a sausage with respect to the ball, i.e., $K = L + B_n$ with $\dim L \leq 1$, then

$$f_{K;B_n}(z) = f_{L+B_n;B_n}(z) = f_{L;B_n}(z+1) = n\mathrm{W}_{n-1}(L)(z+1)^{n-1} + \kappa_n(z+1)^n,$$

because $\mathrm{W}_0(L) = \mathrm{W}_1(L) = \cdots = \mathrm{W}_{n-2}(L) = 0$. Therefore, $-\mathrm{r}(K) = -1$ is an $(n-1)$-fold root of $f_{K;B_n}$. Furthermore, it is easy to check that the quermassintegrals of a sausage $K = L + B_n$ are

$$\mathrm{W}_i(L + B_n) = \frac{n-i}{n}\kappa_{n-1}\mathrm{vol}_1(L) + \kappa_n,$$

and thus one can easily check that

$$\mathrm{D}(K_{-1}) = \mathrm{D}(L) = \mathrm{vol}_1(L) = \frac{n}{\kappa_{n-1}}\bigl(\mathrm{W}_{n-1}(K) - \kappa_n\bigr) = l$$

(see (3.38)). It concludes the proof. □

The following result provides a characterization of sausages with the additional assumption of a lower bound for the mean width of the kernel K_{-1} in terms of the remaining real root of $f_{K;B_n}(z)$.

Theorem 3.3.4 ([62]). *Let $K \in \mathcal{K}^n$ with $\mathrm{r}(K) = 1$. Then K is a sausage with respect to B_n if and only if -1 is an $(n-1)$-fold root of $f_{K;B_n}(z)$ and $\mathrm{b}(K_{-1}) \geq 2l\kappa_{n-1}/(n\kappa_n)$, where l is defined in* (3.36).

Proof. First we suppose that $-\mathrm{r}(K) = -1$ is an $(n-1)$-fold root of the Steiner polynomial $f_{K;B_n}(z)$ and $\mathrm{b}(K_{-1}) \geq 2\,l\kappa_{n-1}/(n\kappa_n)$. By (3.38) we get $\mathrm{W}_{n-1}(K) = \kappa_n + l\kappa_{n-1}/n$, and using the inclusion $K_{-1} + B_n \subset K$, we have

$$\kappa_n + \frac{l\kappa_{n-1}}{n} = \mathrm{W}_{n-1}(K) \geq \mathrm{W}_{n-1}(K_{-1} + B_n) = \frac{\kappa_n}{2}\,\mathrm{b}(K_{-1}) + \kappa_n \geq \frac{l\kappa_{n-1}}{n} + \kappa_n,$$

which implies that $\mathrm{W}_{n-1}(K_{-1}+B_n)=\mathrm{W}_{n-1}(K)$. Hence $K_{-1}+B_n=K$ (see e.g. page 48 in [9]). Now, from the fact that -1 is an $(n-1)$-fold root of $f_{K;B_n}(z)$ and since

$$f_{K;B_n}(z)=f_{K_{-1}+B_n;B_n}(z)=f_{K_{-1};B_n}(z+1)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K_{-1})(z+1)^i,$$

we get that $\mathrm{W}_i(K_{-1})=0$ for all $i=0,\dots,n-2$, which implies that $\dim K_{-1}\leq 1$. Then K is a sausage with respect to the ball.

Conversely, if $K=L+B_n$ with $\dim L\leq 1$, then $-\mathrm{r}(K)=-1$ is an $(n-1)$-fold root of $f_{K;B_n}(z)$ and, moreover, using (3.38),

$$\mathrm{b}(K_{-1})=\mathrm{b}(L)=\frac{2}{\kappa_n}\mathrm{W}_{n-1}(L)=\frac{2}{\kappa_n}\big(\mathrm{W}_{n-1}(K)-\kappa_n\big)=\frac{2}{n}\frac{\kappa_{n-1}}{\kappa_n}\,l. \qquad \square$$

Remark 3.9. *We notice that since the remaining real root γ satisfies*

$$n\mathrm{W}_{n-1}(K)=\kappa_n\big(-\gamma+(n-1)\big)$$

(see (3.36) *and* (3.38)*), then the conditions on the diameter and the mean width in Theorems 3.3.3 and 3.3.4, respectively, can be rewritten in terms of the quermassintegrals of K:*

$$\begin{aligned}\mathrm{D}(K_{-1})&\geq l=-\frac{\kappa_n}{\kappa_{n-1}}(\gamma+1)=\frac{n}{\kappa_{n-1}}\big(\mathrm{W}_{n-1}(K)-\kappa_n\big),\\ \mathrm{b}(K_{-1})&\geq\frac{2}{n}\frac{\kappa_{n-1}}{\kappa_n}\,l=-\frac{2}{n}(\gamma+1)=\frac{2}{\kappa_n}\big(\mathrm{W}_{n-1}(K)-\kappa_n\big).\end{aligned}$$

We observe that the term $\mathrm{W}_{n-1}(K)-\kappa_n\geq 0$ (cf. (1.12) *for $\mathrm{r}(K)=1$).*

3.4 Linearity of the determinant

In this section, we show the characterization of the *linearity of the determinant* (in the same sense as for the volume function $\mathrm{V}_{K;E}$) of positive definite symmetric matrices via 'sausages' of matrices, i.e., the sum of a matrix of rank (at most) 1 and another matrix. We notice that like for $\mathrm{V}_{K;E}$, where for $\lambda\notin[0,1]$ we lose the geometry, for positive definite symmetric matrices, we would lose the positivity if we let λ run outside $[0,1]$.

The Brunn-Minkowski inequality has also its counterpart for matrices. However, conditions for positive definite symmetric matrices A,B to fulfill a result of the type of Theorem 2.1.2 are not known to us. Of course, assumptions on common/equal-volume projection onto a hyperplane (or maximal volume sections through parallel hyperplanes to a given one) of the parallelepipeds whose volume is given by the determinants of A and B are enough (for the volume of the convex combination of those parallelepipeds). Nevertheless it cannot be read in terms of the determinant of $\lambda A+(1-\lambda)B$. For further information on these topics see e.g. [4] and the references inside.

We first prove the following property for diagonal matrices.

Proposition 3.4.1 (Linearity case for orthogonal boxes, [50]). *Let* $A, B \in \mathbb{R}^{n\times n}$ *be diagonal matrices. Then*

$$\det\big(\lambda A + (1-\lambda)B\big) = \lambda \det A + (1-\lambda)\det B,$$

if and only if $B = L + A$*, where* L *is a diagonal matrix such that* $\operatorname{rank} L \leq 1$.

Proof. Let $B = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and $A = \operatorname{diag}(\lambda_1 + \varepsilon_1, \dots, \lambda_n + \varepsilon_n)$ with $\varepsilon_i \in \mathbb{R}$ for all $i = 1, \dots, n$. Then, for all $\lambda \in [0,1]$, we have

$$\prod_{i=1}^{n}(\varepsilon_i\lambda + \lambda_i) = \det\big(\lambda A + (1-\lambda)B\big) = \lambda \det A + (1-\lambda)\det B = \prod_{i=1}^{n}\lambda_i + \lambda\left(\prod_{i=1}^{n}(\varepsilon_i + \lambda_i) - \prod_{i=1}^{n}\lambda_i\right).$$

Identifying the coefficients of both polynomials in λ, we get that the set $\{1 \leq i \leq n : \varepsilon_i \neq 0\}$ has at most one element, which implies that at least $n-1$ of the ε_i's vanish. It concludes the proof. □

From the above proposition, next result follows immediately.

Corollary 3.4.1 ([50]). *Let* $K, E \in \mathcal{K}^n$ *be orthogonal boxes. Then*

$$\operatorname{vol}\big(\lambda K + (1-\lambda)E\big) = \lambda \operatorname{vol}(K) + (1-\lambda)\operatorname{vol}(E),$$

if and only if the pair K, E *is a sausage.*

Theorem 3.4.1 ([50]). *Let* $A, B \in \mathbb{R}^{n\times n}$ *be positive definite (symmetric) matrices. Then*

$$\det\big(\lambda A + (1-\lambda)B\big) = \lambda \det A + (1-\lambda)\det B, \tag{3.39}$$

if and only if $B = L + A$*, with* $\operatorname{rank} L \leq 1$.

Proof. First we assume that condition (3.39) holds. Let $T \in \mathbb{R}^{n\times n}$ be an orthogonal matrix such that $T^\intercal A T = \operatorname{diag}(a_1, \dots, a_n)$, where $a_i > 0$ are the eigenvalues of the matrix A. Denoting by $\widetilde{T} = T \operatorname{diag}\big(1/\sqrt{a_1}, \dots, 1/\sqrt{a_n}\big)$ we get that $\widetilde{T}^\intercal A \widetilde{T} = \mathrm{I}_n$. Since $\widetilde{T}^\intercal B \widetilde{T}$ is positive definite and symmetric, there exists an orthogonal matrix $S \in \mathbb{R}^{n\times n}$ such that $S^\intercal \widetilde{T}^\intercal B \widetilde{T} S = \operatorname{diag}(y_1, \dots, y_n)$, with $y_i > 0$, and then

$$\prod_{i=1}^{n} y_i = \det B \det \widetilde{T}^\intercal \det \widetilde{T} = \frac{\det B}{\det A}.$$

Therefore

$$S^\intercal \widetilde{T}^\intercal \big(\lambda A + (1-\lambda)B\big)\widetilde{T} S = \operatorname{diag}\big(\lambda + (1-\lambda)y_1, \dots, \lambda + (1-\lambda)y_n\big)$$

and hence, using (3.39),

$$\begin{aligned}
\det\big(\lambda \mathrm{I}_n + (1-\lambda)\operatorname{diag}(y_1, \dots, y_n)\big) &= \det\Big(S^\intercal \widetilde{T}^\intercal \big(\lambda A + (1-\lambda)B\big)\widetilde{T} S\Big) \\
&= \frac{1}{\det A}\det\big(\lambda A + (1-\lambda)B\big) \\
&= \lambda + (1-\lambda)\frac{\det B}{\det A} \\
&= \lambda \det \mathrm{I}_n + (1-\lambda)\det \operatorname{diag}(y_1, \dots, y_n).
\end{aligned}$$

From the linearity for the determinant of diagonal matrices (see Proposition 3.4.1), we have that $\operatorname{diag}(y_1, \ldots, y_n) = L_1 + \mathrm{I}_n$, with $\operatorname{rank} L_1 \leq 1$, or equivalently $S^{\intercal}\widetilde{T}^{\intercal}B\widetilde{T}S = L_1 + S^{\intercal}\widetilde{T}^{\intercal}A\widetilde{T}S$. So, it follows that $B = PL_1Q + A$ where P and Q are invertible matrices, which implies that $L = PL_1Q$ has rank at most 1.

The converse can be shown in a similar way by bringing A and B to a diagonal form and using Proposition 3.4.1. □

Chapter 4

On the roots of generalized Wills μ-polynomials and m-polynomials

In the previous chapter, we studied the problem of recovering sausages via some conditions on the roots of Steiner polynomials. Moreover, these roots turned out to be a natural tool in order to deal with the equality cases of some linear refinements of the Brunn-Minkowski inequality as well as other related topics.

Because of this fact, and motivated by previous works of Henk, Hernández Cifre and Saorín Gómez, on the roots of the Steiner polynomial (see [26, 27]), in this chapter we are interested in studying properties of the roots of a more general family of geometric polynomials of convex bodies (extending the Steiner polynomial). Moreover, we will mainly focus on those which have a strong connection with the well-known Wills functional; see Section 1.2 and the references there for further information about this topic.

We notice also that the (relative) Wills polynomial $\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)/\kappa_i z^i$ (cf. (1.10)) can be seen as a Steiner polynomial with certain 'weights'. This fact, together with the (previously commented) goal of the chapter, has lead us to consider the following definition:

Definition 4.1. *Let $\mathbf{m} = (m_i)_{i\in\mathbb{N}}$ be a sequence of positive real numbers and let $K, E \in \mathcal{K}^n$ be convex bodies with $\dim(K+E) = n$. Then the* $\mathbf{m}$-polynomial *of K and E is the formal polynomial, in the complex variable $z \in \mathbb{C}$,*

$$f_{K;E}^{\mathbf{m}}(z) = \sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\,m_i z^i.$$

If the weights m_i are the moments of some measure μ on the non-negative real line $\mathbb{R}_{\geq 0}$, i.e., if

$$m_i = m_i(\mu) = \int_0^\infty t^i \,\mathrm{d}\mu(t), \quad i = 0, \dots, n,$$

then it can be shown that the corresponding $\mathbf{m}$-polynomial comes from the natural generalization of the Wills functional/polynomial. This fact is the starting point of the following section, where it will be carefully shown. The original work that we collect in this chapter can be found in [29, 30, 63].

4.1 On the roots of generalized Wills μ-polynomials

4.1.1 On the Wills type functionals

Recently, Kampf [32] has proved that generalizations of relations (1.11) remain true when the 'distance' $d_E(x, K)$, between $x \in \mathbb{R}^n$ and K, relative to a convex body E with $0 \in \operatorname{int} E$ is considered, i.e.,

$$\int_{\mathbb{R}^n} e^{-\pi d_E(x,K)^2} \mathrm{d}x = 2\pi \int_0^\infty \operatorname{vol}(K + tE) t\, e^{-\pi t^2} \mathrm{d}t = \sum_{i=0}^n \binom{n}{i} \frac{\mathrm{W}_i(K; E)}{\kappa_i}.$$

Moreover, a more general functional can be obtained replacing $e^{-\pi t^2}$ by a function $G(t)$ which is properly associated to a measure μ on the non-negative real line $\mathbb{R}_{\geq 0}$:

$$\int_{\mathbb{R}^n} G\big(d_E(x, K)\big) \mathrm{d}x \quad \text{with} \quad G(t) = \mu\big([t, \infty)\big). \tag{4.1}$$

We extend this functional to any pair of convex bodies K, E, allowing gauge bodies E with dimension $\dim E < n$.

To this aim, for a given $E \in \mathcal{K}^n$ with $0 \in \operatorname{relint} E$ and $x - y \in \operatorname{lin} E$, let

$$d_E(x, y) = \inf\{\lambda \geq 0 : \, x - y \in \lambda E\}.$$

We notice that if E is 0-symmetric, i.e., $E = -E$, this function defines a distance on $\operatorname{lin} E$. Then, for $x \in K + \operatorname{lin} E$, we have

$$d_E(x, K) = \inf\big\{d_E(x, y) : y \in K \cap (x + \operatorname{lin} E)\big\} = \inf\{r \geq 0 : x \in K + rE\}.$$

Thus, the expression $d_E(x, K)$ is only defined for $x \in K + \operatorname{lin} E$ and, following the idea used in [32], the next result is fulfilled.

Lemma 4.1.1 ([29]). *Let μ be a finite measure on $\mathbb{R}_{\geq 0}$ such that the moments $m_i(\mu) = \int_0^\infty t^i \,\mathrm{d}\mu(t)$ of μ, $i = 0, \dots, n$, exist and are finite, and let $G(t) = \mu\big([t, \infty)\big)$, $t \in \mathbb{R}_{\geq 0}$. Then, for $K, E \in \mathcal{K}^n$ with $0 \in \operatorname{relint} E$, we have that*

$$\int_{K + \operatorname{lin} E} G\big(d_E(x, K)\big) \,\mathrm{d}x = \int_0^\infty \operatorname{vol}(K + tE) \,\mathrm{d}\mu(t) = \sum_{i=0}^n \binom{n}{i} \mathrm{W}_i(K; E) m_i(\mu).$$

We follow the idea of the proof given by Kampf in [32], which is based on Fubini's theorem (see Theorem 1.2.1). Here χ_M will denote the characteristic function of the set $M \subset \mathbb{R}^n$.

Proof. Clearly, for $x \in K + \operatorname{lin} E$ we have $d_E(x,K) \le t$ if and only if $x \in K + tE$. Using this property and Steiner formula (1.5), we get

$$\begin{aligned}\int_{K+\operatorname{lin} E} G\big(d_E(x,K)\big)\,\mathrm{d}x &= \int_{K+\operatorname{lin} E} \mu\big([d_E(x,K),\infty)\big)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^n}\int_0^\infty \chi_{\{y\in K+\operatorname{lin} E:\, d_E(y,K)\le t\}}(x)\,\mathrm{d}\mu(t)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^n}\int_0^\infty \chi_{K+tE}(x)\,\mathrm{d}\mu(t)\,\mathrm{d}x = \int_0^\infty\int_{\mathbb{R}^n} \chi_{K+tE}(x)\,\mathrm{d}x\,\mathrm{d}\mu(t)\\ &= \int_0^\infty \operatorname{vol}(K+tE)\,\mathrm{d}\mu(t) = \sum_{i=0}^n \binom{n}{i} \mathrm{W}_i(K;E)\int_0^\infty t^i\,\mathrm{d}\mu(t)\\ &= \sum_{i=0}^n \binom{n}{i} \mathrm{W}_i(K;E)\, m_i(\mu).\end{aligned}$$

It shows the lemma. □

We observe that the right-hand side in the last equality is translation invariant. Thus, for any convex bodies $K, E \in \mathcal{K}^n$, any $x_0 \in \operatorname{relint} E$, and a given measure μ on $\mathbb{R}_{\ge 0}$, we have a Wills type functional (associated to μ)

$$\mathcal{W}^\mu(K;E) = \int_{K+\operatorname{lin}(E-x_0)} G\big(d_{E-x_0}(x,K)\big)\,\mathrm{d}x = \sum_{i=0}^n \binom{n}{i}\mathrm{W}_i(K;E) m_i(\mu).$$

Thus, we can always assume, without loss of generality, that $0 \in \operatorname{relint} E$.

Using the previous lemma for the function $G(t) = e^{-\pi t^2}$, we get the *relative Wills functional* for convex bodies $K, E \in \mathcal{K}^n$, namely,

$$\mathcal{W}(K;E) = \int_{K+\operatorname{lin} E} e^{-\pi d_E(x,K)^2}\mathrm{d}x = \sum_{i=0}^n \binom{n}{i}\frac{\mathrm{W}_i(K;E)}{\kappa_i}. \tag{4.2}$$

In the case of the Steiner functional $\sum_{i=0}^n \binom{n}{i}\mathrm{W}_i(K;E)$, we show that it can be also obtained as a generalized Wills type functional for a particular 'limit' measure. This is the content of the following result.

Theorem 4.1.1 ([29]). *Let $K, E \in \mathcal{K}^n$ with $0 \in \operatorname{relint} E$. Then*

$$\sum_{i=0}^n \binom{n}{i}\mathrm{W}_i(K;E) = \lim_{\sigma\to 0^+}\int_{K+\operatorname{lin} E}\int_{d_E(x,K)}^\infty \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{(t-1)^2}{2\sigma^2}}\,\mathrm{d}t\,\mathrm{d}x.$$

Moreover, such an expression (in which a non-discrete *measure μ on $\mathbb{R}_{\ge 0}$ is considered) is only possible through a 'pass to the limit' process.*

Proof. Let μ_σ be the measure on $\mathbb{R}_{\geq 0}$ given by

$$\mu_\sigma(A) = \int_A f_\sigma(t)\,\mathrm{d}t, \qquad \text{with} \quad f_\sigma(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-1)^2}{2\sigma^2}},$$

(see Figure 4.1) and let G_σ be the function

$$G_\sigma(s) = \int_s^\infty \mathrm{d}\mu_\sigma(t) = \int_s^\infty f_\sigma(t)\,\mathrm{d}t.$$

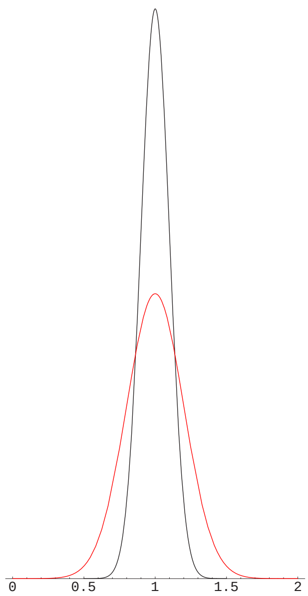

Figure 4.1: The function $f_\sigma(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-1)^2}{2\sigma^2}}$ for $\sigma = 0.2$ (red) and $\sigma = 0.1$.

On the one hand we observe that, denoting by $\overline{\mu}_\sigma$ the measure on $\mathbb{R}_{\leq 0}$ associated to the function $f_\sigma(t)$, we have

$$\begin{aligned}
\lim_{\sigma\to 0^+} \int_{-\infty}^0 t^r\,\mathrm{d}\overline{\mu}_\sigma(t) &= \lim_{\sigma\to 0^+} \int_{-\infty}^0 t^r f_\sigma(t)\,\mathrm{d}t \\
&= \lim_{\sigma\to 0^+} \int_{-\infty}^{-1/(\sqrt{2}\sigma)} \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\sigma s + 1\right)^r e^{-s^2}\mathrm{d}s \\
&= \lim_{\sigma\to 0^+} \int_{-\infty}^{-1/(\sqrt{2}\sigma)} \frac{1}{\sqrt{\pi}} \left(\sum_{i=0}^r \binom{r}{i} \left(\sqrt{2}\sigma s\right)^i\right) e^{-s^2}\mathrm{d}s \\
&= \frac{1}{\sqrt{\pi}} \sum_{i=0}^r \binom{r}{i} 2^{i/2} \left(\lim_{\sigma\to 0^+} \sigma^i \int_{-\infty}^{-1/(\sqrt{2}\sigma)} s^i e^{-s^2}\,\mathrm{d}s\right) = 0.
\end{aligned}$$

On the other hand, if $\varphi_\sigma(t) = e^{t+t^2\sigma^2/2}$ denotes the moment generating function of a normal distribution $\mathrm{N}(1,\sigma)$ associated to the density function f_σ (on the real line $\mathbb{R}$), then its i-th derivative can be written as

$$\varphi_\sigma^{(i)}(t) = e^{t+t^2\sigma^2/2}\left(1 + t\sigma^2\right)^i + \sigma^2 g_{\sigma;i}(t),$$

where $g_{\sigma;i}(t)$ is given by the inductive formula

$$g_{\sigma;0}(t)=0, \quad g_{\sigma;i+1}(t)=g'_{\sigma;i}(t)+\frac{e^{t+t^2\sigma^2/2}}{\sigma^2}\left(\left(1+t\sigma^2\right)^i\right)'.$$

We observe that $g_{\sigma;i}(0)$ is bounded in a neighborhood of $\sigma=0$, and thus $\lim_{\sigma\to 0^+}\varphi_\sigma^{(i)}(0)=1$. So,

$$\int_{K+\operatorname{lin}E} G_\sigma\big(d_E(x,K)\big)\,\mathrm{d}x=\sum_{i=0}^n\binom{n}{i}\mathrm{W}_i(K;E)\int_0^\infty t^i\,\mathrm{d}\mu_\sigma(t)$$

by Lemma 4.1.1, and then

$$\begin{aligned}\lim_{\sigma\to 0^+}\int_{K+\operatorname{lin}E} G_\sigma\big(d_E(x,K)\big)\,\mathrm{d}x&=\sum_{i=0}^n\binom{n}{i}\mathrm{W}_i(K;E)\lim_{\sigma\to 0^+}\int_0^\infty t^i\,\mathrm{d}\mu_\sigma(t)\\&=\sum_{i=0}^n\binom{n}{i}\mathrm{W}_i(K;E)\left(\lim_{\sigma\to 0^+}\int_0^\infty t^i\,\mathrm{d}\mu_\sigma(t)+\lim_{\sigma\to 0^+}\int_{-\infty}^0 t^i\,\mathrm{d}\overline{\mu}_\sigma(t)\right)\\&=\sum_{i=0}^n\binom{n}{i}\mathrm{W}_i(K;E)\lim_{\sigma\to 0^+}\int_{-\infty}^\infty t^i\,f_\sigma(t)\mathrm{d}t\\&=\sum_{i=0}^n\binom{n}{i}\mathrm{W}_i(K;E)\lim_{\sigma\to 0^+}\varphi_\sigma^{(i)}(0)=\sum_{i=0}^n\binom{n}{i}\mathrm{W}_i(K;E),\end{aligned}$$

where, in the last equality but one, we use the well known connection between the moments of a measure and its moment generating function (see e.g. Sections 2.3–2.4 and Theorem 2.3.7 in [11]).

Finally, the last assertion of the theorem follows from the fact that if, for a measure $\widetilde{\mu}$, $m_i(\widetilde{\mu})=1$ holds for all $i=0,\dots,n$, with $n\geq 2$, then

$$\int_0^\infty (t-1)^2\,\mathrm{d}\widetilde{\mu}(t)=\int_0^\infty t^2\,\mathrm{d}\widetilde{\mu}(t)-2\int_0^\infty t\,\mathrm{d}\widetilde{\mu}(t)+\int_0^\infty \mathrm{d}\widetilde{\mu}(t)=0,$$

which implies that $\widetilde{\mu}$ is a discrete measure concentrated at $t=1$. □

4.1.2 The cone of roots of Wills μ-polynomials

Definition 4.2. *Let $K,E\in\mathcal{K}^n$ and let μ be a measure on $\mathbb{R}_{\geq 0}$ with finite moments $m_i(\mu)$, $i=0,\dots,n$. Then*

$$f_{K;E}^\mu(z)=\sum_{i=0}^n\binom{n}{i}\mathrm{W}_i(K;E)m_i(\mu)z^i$$

will denote the Wills μ-polynomial *of K with respect to E, regarded as a formal polynomial in a complex variable $z\in\mathbb{C}$.*

Similarly, we will represent the relative Steiner and Wills polynomial in a variable $z\in\mathbb{C}$ (cf. (1.10), (4.2)), respectively, by

$$f_{K;E}(z)=\sum_{i=0}^n\binom{n}{i}\mathrm{W}_i(K;E)z^i \quad\text{and}\quad f_{K;E}^g(z)=\sum_{i=0}^n\binom{n}{i}\frac{\mathrm{W}_i(K;E)}{\kappa_i}z^i.$$

From now on, we will denote by g the measure associated to $G(t) = e^{-\pi t^2}$, which yields the (classical) Wills functional when $E = B_n$ (cf. (1.11)), and whose moments are $m_i(g) = 1/\kappa_i$.

Here we are interested in studying properties of the roots of the above family of polynomials $f^{\mu}_{K;E}(z)$. To this end, we fix the notation which will be used along the chapter. We will denote by $\operatorname{Re} z$, $\operatorname{Im} z$ and $\arg z$, the real part, imaginary part and the principal argument of a complex number z, respectively, whereas $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$ will be the set of complex numbers with non-negative imaginary part. In order to establish most of the results contained in this chapter involving the roots of $f^{\mu}_{K;E}(z)$, we will need the following definition.

Definition 4.3. *Given a measure μ on $\mathbb{R}_{\geq 0}$, let*

$$\mathcal{R}^{\mu}(n) = \left\{z \in \mathbb{C}^+ : f^{\mu}_{K;E}(z) = 0 \text{ for } K, E \in \mathcal{K}^n, \dim(K + E) = n\right\}$$

be the set of all roots of $f^{\mu}_{K;E}(z)$, $K, E \in \mathcal{K}^n$, in the upper half-plane.

We notice that the condition $\dim(K + E) = n$ in the above definition is needed in order to avoid identically 0 polynomials (cf. Proposition 1.2.1 (v) and Steiner formula (1.5)).

From now on and unless we explicitly say the opposite, we will always assume that, for a given measure μ on $\mathbb{R}_{\geq 0}$, its moments $m_i(\mu) > 0$ for all $i \geq 0$, i.e., we omit the case when μ is discrete and concentrates the measure at $t = 0$. We will also need the following additional notation. For convex bodies $K, E \in \mathcal{K}^n$ such that $f^{\mu}_{K;E}(z)$ has a non-zero root let

$$\theta^{\mu}_{K;E} = \min\left\{\arg z : z \in \mathbb{C}^+ \backslash \{0\},\, f^{\mu}_{K;E}(z) = 0\right\}, \tag{4.3}$$

(see Figure 4.2) and we denote by

$$\mathcal{R}^{\mu}(K; E) = \left\{z \in \mathbb{C}^+ \backslash \{0\} : \arg z \geq \theta^{\mu}_{K;E}\right\} \cup \{0\} \tag{4.4}$$

the convex cone, in the upper half-plane, generated as the positive hull of the roots of the polynomial $f^{\mu}_{K;E}(z)$ and $\mathbb{R}_{\leq 0}$.

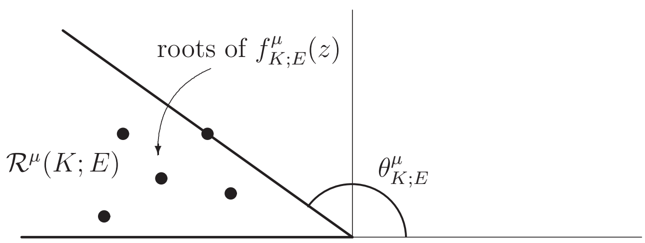

Figure 4.2: The angle $\theta^{\mu}_{K;E}$ and the cone $\mathcal{R}^{\mu}(K; E)$.

Along this chapter we will need several different sets of roots of polynomials. Next we collect all of them.

For a fixed gauge body $E \in \mathcal{K}^n$ we define

$$\mathcal{R}^\mu(n;E) = \bigl\{z \in \mathbb{C}^+ : f^\mu_{K;E}(z) = 0 \text{ for } K \in \mathcal{K}^n,\, \dim(K+E) = n\bigr\}. \tag{4.5}$$

The set of roots of all Steiner polynomials $f_{K;E}(z)$, $K, E \in \mathcal{K}^n$, in the upper half-plane will be denoted by

$$\mathcal{R}(n) = \bigl\{z \in \mathbb{C}^+ : f_{K;E}(z) = 0 \text{ for } K, E \in \mathcal{K}^n,\, \dim(K+E) = n\bigr\}. \tag{4.6}$$

Finally, in the analogous way to (4.5), we denote by

$$\mathcal{R}(n;E) = \bigl\{z \in \mathbb{C}^+ : f_{K;E}(z) = 0 \text{ for } K \in \mathcal{K}^n,\, \dim(K+E) = n\bigr\}. \tag{4.7}$$

We start stating a preliminary lemma which will be needed for the proof of Theorem 4.1.3. In Theorem 5.2 of [34] the following result is proved.

Theorem 4.1.2 (Kato, [34]). *Let $\xi(t)$ be an unordered n-tuple of complex numbers, depending continuously on a real variable t in a (closed or open) interval I. Then there exist n continuous functions $\nu_i(t)$, $i = 1, \dots, n$, which constitute the values of the n-tuple $\xi(t)$ for each $t \in I$.*

As a consequence, we get the following lemma.

Lemma 4.1.2 ([29]). *Let $K(t) \in \mathcal{K}^n$, $t \in [a,b]$, be a one-parameter continuous (on t) family of convex bodies with $\dim K(a) = n-k-1$ and $\dim K(t) = n-k$ for all $t \in (a,b]$, and let $E \in \mathcal{K}^n$ with $\dim E = r > k$ and $\dim\bigl(K(t)+E\bigr) = n$ for all $t \in [a,b]$. Let $f^\mu_{K(t);E}(z)$, $t \in [a,b]$, be the corresponding one-parameter family of μ-polynomials. Then:*

- *i) There exist $r-k-1$ continuous functions $\nu_1, \dots, \nu_{r-k-1} : [a,b] \longrightarrow \mathbb{C}$ joining the $r-k-1$ non-zero roots of $f^\mu_{K(a);E}(z)$ and $r-k-1$ non-zero roots of $f^\mu_{K(b);E}(z)$, such that $\nu_1(t), \dots, \nu_{r-k-1}(t)$ are $r-k-1$ of the $r-k$ non-zero roots of $f^\mu_{K(t);E}(z)$ for all $t \in [a,b]$.*
- *ii) Moreover, there exists another continuous function $\nu_{r-k} : (a,b] \longrightarrow \mathbb{C}$ such that $\nu_{r-k}(t)$ is the remaining root of $f^\mu_{K(t);E}(z)$ for all $t \in (a,b]$, satisfying that $\lim_{t\to a^+} \nu_{r-k}(t) = 0$.*

Proof. Since

$$c(t) = \binom{n}{r} \mathrm{W}_r\bigl(K(t);E\bigr) m_r(\mu) \neq 0$$

for all $t \in [a,b]$, the result is a direct consequence of Theorem 4.1.2 and the fact that the roots of a polynomial are continuous functions of the coefficients (see Theorem 1.5.1) applied to the polynomials

$$\frac{f^\mu_{K(t);E}(z)}{c(t) z^k} = \frac{1}{c(t)} \sum_{i=0}^{r-k} \binom{n}{k+i} \mathrm{W}_{k+i}\bigl(K(t);E\bigr) m_{k+i}(\mu) z^i,$$

whose leading coefficients are 1 for all $t \in [a,b]$. □

We start showing that the set of roots in the upper half-plane is a convex cone for any measure μ on $\mathbb{R}_{\geq 0}$. In fact, as it will be commented later on, from the proof we may assert that the following result holds even in the most general case of $\mathbf{m}$-polynomials (it is only needed that $m_i > 0$ for all $i = 0, \dots, n$).

Theorem 4.1.3 ([29]). *For any measure μ, $\mathcal{R}^\mu(n)$ is a convex cone, containing the non-positive real axis $\mathbb{R}_{\leq 0}$.*

Proof. By the homogeneity of the quermassintegrals we have that for convex bodies $K, E \in \mathcal{K}^n$ and $\lambda > 0$, $f^\mu_{\lambda K;E}(\lambda z) = \lambda^n f^\mu_{K;E}(z)$. Hence, if $\nu \in \mathcal{R}^\mu(n)$, $\nu \neq 0$, then there exist $K, E \in \mathcal{K}^n$ such that $f^\mu_{K;E}(\nu) = 0$ and thus, for each $\lambda > 0$, $0 = f^\mu_{K;E}(\nu) = f^\mu_{\lambda K;E}(\lambda\nu)/\lambda^n$. Therefore $\lambda\nu \in \mathcal{R}^\mu(n)$ and so, $\mathcal{R}^\mu(n)$ is a cone.

In order to prove the convexity of $\mathcal{R}^\mu(n)$ it suffices to show that for any $\nu_0 \in \mathcal{R}^\mu(n)$ fixed, $\nu_0 \neq 0$, the cone

$$\mathcal{R}^\mu(n) \cap \Big(\big\{z \in \mathbb{C}^+ \backslash \{0\} : \arg z \geq \arg \nu_0\big\} \cup \{0\}\Big)$$

is convex. To this end, let $K, E \in \mathcal{K}^n$ be such that $f^\mu_{K;E}(\nu_0) = 0$. Without loss of generality we may assume that $\operatorname{aff} E = \big\{(x_1, \dots, x_n)^\intercal \in \mathbb{R}^n : x_1 = \dots = x_{n-r} = 0\big\}$, where $r = \dim E$. Let $H_i = \big\{(x_1, \dots, x_n)^\intercal \in \mathbb{R}^n : x_{n-r+1} = \dots = x_{n-r+i} = 0\big\}$, $i = 1, \dots, r-1$, be the $(n-i)$-dimensional coordinate plane containing $(\operatorname{aff} E)^\perp$, and let $K_i = K|H_i$, $i = 1, \dots, r-1$, with $K_0 = K$.

With this notation we will show, by finite induction on $j = r - i$, with $j = 1, \dots, r$, that all the points of $\mathcal{R}^\mu(K_i; E)$ (cf. (4.4)) are roots of some μ-polynomial, i.e., that $\mathcal{R}^\mu(K_i; E) \subset \mathcal{R}^\mu(n)$. So $\mathcal{R}^\mu(K; E) \subset \mathcal{R}^\mu(n)$, which will show the convexity of $\mathcal{R}^\mu(n) \cap \big(\{z \in \mathbb{C}^+ \backslash \{0\} : \arg z \geq \arg \nu_0\} \cup \{0\}\big)$.

If $j = 1$, then the polynomial $f^\mu_{K_{r-1};E}(z)$ reduces to

$$\left[\binom{n}{r-1} \mathrm{W}_{r-1}(K_{r-1}; E)\, m_{r-1}(\mu) + \binom{n}{r} \mathrm{W}_r(K_{r-1}; E)\, m_r(\mu) z\right] z^{r-1},$$

and so it has only a non-zero real root. Thus, $\mathbb{R}_{\leq 0} = \mathcal{R}^\mu(K_{r-1}; E) \subset \mathcal{R}^\mu(n)$ and in particular, we have that $\mathcal{R}^\mu(n)$ contains the non-positive real axis.

Now we assume $1 < j \leq r$ and that the result is true for $j - 1$, i.e., we suppose that we have $\mathcal{R}^\mu(K_{r-j+1}; E) \subset \mathcal{R}^\mu(n)$. We notice that the strict inclusion $\mathcal{R}^\mu(K_{r-j+1}; E) \subsetneq \mathcal{R}^\mu(K_{r-j}; E)$ can be assumed, otherwise the required result is directly obtained. For each $t \in [0, 1]$, we consider the convex body

$$K(t) = tK_{r-j+1} + (1 - t)K_{r-j},$$

and let ν_j be a root of the polynomial $f^\mu_{K_{r-j};E}(z)$ such that $\arg \nu_j = \theta^\mu_{K_{r-j};E}$. The family of sets $K(t)$, $t \in [0, 1]$, provides a one-parameter family of μ-polynomials $f^\mu_{K(t);E}(z)$ satisfying the conditions of Lemma 4.1.2, and hence there exists a continuous map $\nu : [0, 1] \longrightarrow \mathbb{C}$ with $\nu(0) = \nu_j$ and $\nu(1) = \nu_{j-1}$ being a root of $f^\mu_{K_{r-j+1};E}(z)$, such that $\nu(t)$ is a root of $f^\mu_{K(t);E}(z)$ for all $t \in [0, 1]$.

Without loss of generality we may assume that ν_j is not the root which 'goes to zero'; otherwise, we can work with its conjugate $\overline{\nu}_j$.

Therefore, $f : [0,1] \longrightarrow (0, 2\pi)$, given by $f(t) = \arg \nu(t)$, is a continuous function satisfying $f(1) = \arg \nu_{j-1} \geq \theta^{\mu}_{K_{r-j+1};E}$ and $f(0) = \theta^{\mu}_{K_{r-j};E}$. Thus, using the intermediate value theorem, together with the fact that $\mathcal{R}^{\mu}(n)$ is a cone and the induction hypothesis, we may conclude that $\mathcal{R}^{\mu}(K_{r-j}; E) \subset \mathcal{R}^{\mu}(n)$. □

We notice that in order to construct these Wills type functionals, we work with a *measure* μ on $\mathbb{R}_{\geq 0}$ (cf. Lemma 4.1.1). The results by Kampf [32] are stated in the more general setting when a *signed measure* ρ is considered. Next we show that in this case the corresponding set $\mathcal{R}^{\rho}(n)$, although it is always a cone (see the proof of Theorem 4.1.3), it is not, in general, convex.

Proposition 4.1.1 ([29]). *There exist signed measures ρ on $\mathbb{R}_{\geq 0}$ such that the cone $\mathcal{R}^{\rho}(n)$ is not convex.*

Proof. Let ρ be the signed measure on $\mathbb{R}_{\geq 0}$ given by

$$\rho(\{0\}) = \frac{7}{6}, \quad \rho(\{1\}) = -\frac{1}{3}, \quad \rho(\{2\}) = \frac{1}{6}, \quad \rho(\mathbb{R}_{\geq 0} \setminus \{0,1,2\}) = 0.$$

Then the first four moments of ρ are given by

$$\begin{pmatrix} m_0(\rho) \\ m_1(\rho) \\ m_2(\rho) \\ m_3(\rho) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 1 & 8 \end{pmatrix} \begin{pmatrix} 7/6 \\ -1/3 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1/3 \\ 1 \end{pmatrix}.$$

Hence, in dimension $n = 3$, any ρ-polynomial takes the form

$$f^{\rho}_{K;E}(z) = \mathrm{W}_0(K;E) + \mathrm{W}_2(K;E)z^2 + \mathrm{W}_3(K;E)z^3.$$

On the one hand, if $K, E \in \mathcal{K}^n$ are convex bodies with $1 \leq \dim K \leq 2$ and $\dim E = 3$, i.e., such that $\mathrm{W}_0(K;E) = 0$ and both $\mathrm{W}_2(K;E)$, $\mathrm{W}_3(K;E) \neq 0$, then $f^{\rho}_{K;E}(z)$ has only non-positive real roots. On the other hand, if $K, E \in \mathcal{K}^n$ have $\dim K = 3$ and $\dim E = 2$, i.e., they are such that $\mathrm{W}_3(K;E) = 0$ and both $\mathrm{W}_0(K;E), \mathrm{W}_2(K;E) \neq 0$, then $f^{\rho}_{K;E}(z)$ has imaginary pure complex roots. Thus

$$\mathbb{R}_{\leq 0} \cup \{r\mathrm{i} \in \mathbb{C} : r \geq 0\} \subset \mathcal{R}^{\rho}(3) \cap \{z \in \mathbb{C}^+ : \operatorname{Re} z \leq 0\}.$$

And moreover, it is easy to check that this inclusion is an equality: indeed, if there exist convex bodies $K, E \in \mathcal{K}^n$ such that

$$f^{\rho}_{K;E}(z) = \mathrm{W}_3(K;E)(z + 1 + b\mathrm{i})(z + 1 - b\mathrm{i})(z + c)$$

for some $b, c \geq 0$, with $\mathrm{W}_3(K;E) \neq 0$, then we get, in particular, the relation

$$\mathrm{W}_3(K;E)(b^2 + 2c + 1) = 0,$$

which is a contradiction. This shows that the cone $\mathcal{R}^{\rho}(3)$ is not convex. □

Characterizing μ-polynomials. Properties of $\mathcal{R}^\mu(n)$.

The main ingredient for most of the following proofs are the well-known Aleksandrov-Fenchel inequalities (1.20) and (1.21). Strongly connected with these inequalities, a sequence of real numbers $a_0,\dots,a_n \geq 0$ is called *ultra-logconcave* if

$$c_{i,n}a_i^2 \geq a_{i-1}a_{i+1}, \qquad \text{with } c_{i,n} = \frac{\binom{n}{i-1}\binom{n}{i+1}}{\binom{n}{i}^2},$$

$1 \leq i \leq n-1$. The following result shows that this property for real numbers allows essentially to characterize Steiner polynomials. It can be found in Lemma 2.1 of [27].

Lemma 4.1.3 (Henk et al., [27]). *A real polynomial $\sum_{i=0}^n a_i z^i$, $a_i \geq 0$, is a Steiner polynomial $f_{K;E}(z)$ for a pair of convex bodies $K,E \in \mathcal{K}^n$, with $\dim K = n-k$, $\dim E = r$, $\dim(K+E) = n$, if and only if*

i) $a_i > 0$ for all $k \leq i \leq r$, and $a_i = 0$ otherwise, and

ii) the sequence $a_0,\dots,a_n$ is ultra-logconcave.

This result substantially follows from Shephard's Theorem 1.4.2, but here the number of involved inequalities is reduced and the construction of the two convex bodies is extended to the more general case $\mathrm{W}_i \geq 0$. Moreover, the above lemma can be rewritten in terms of the roots of the Steiner polynomial; this is the content of the following result which can be found in Corollary 2.1 of [27]. In order to state it, we need additional notation.

For complex numbers $z_1,\dots,z_r \in \mathbb{C}$ let

$$\mathrm{s}_i\big(z_1,\dots,z_r\big) = \sum_{\substack{J\subset\{1,\dots,r\}\\ \#J=i}} \prod_{j\in J} z_j$$

denote the i-th elementary symmetric function of $z_1,\dots,z_r$, $1 \leq i \leq r$, setting $\mathrm{s}_0\big(z_1,\dots,z_r\big) = 1$.

Corollary 4.1.1 (Henk et al., [27]). *The complex numbers $\gamma_1,\dots,\gamma_r \in \mathbb{C}$ are the roots of a Steiner polynomial $f_{K;E}(z)$ of degree $r \leq n$, with $K,E \in \mathcal{K}^n$, $\dim K = n-k$, $\dim E = r$ and $\dim(K+E) = n$, if and only if*

$$\begin{aligned} \text{i)}\quad & (-1)^i\mathrm{s}_i\big(\gamma_1,\dots,\gamma_r\big) > 0, \quad 0 \leq i \leq r-k,\\ & \mathrm{s}_i\big(\gamma_1,\dots,\gamma_r\big) = 0, \quad r-k+1 \leq i \leq r,\\ \text{ii)}\quad & c_{r-i,n}\,\mathrm{s}_i\big(\gamma_1,\dots,\gamma_r\big)^2 \geq \mathrm{s}_{i-1}\big(\gamma_1,\dots,\gamma_r\big)\mathrm{s}_{i+1}\big(\gamma_1,\dots,\gamma_r\big),\ 1 \leq i \leq r-1. \end{aligned}$$

The above results (Lemma 4.1.3 and Corollary 4.1.1) can be also exploited to characterize μ-polynomials for a given measure μ. Indeed, setting

$$c_{i,n}^\mu = \frac{\binom{n}{i-1}m_{i-1}(\mu)\binom{n}{i+1}m_{i+1}(\mu)}{\binom{n}{i}^2 m_i(\mu)^2}$$

and, since a (real) polynomial $\sum_{i=0}^n a_i z^i$, $a_i \geq 0$, is a μ-polynomial if and only if $\sum_{i=0}^n \big(a_i/m_i(\mu)\big) z^i$ is a Steiner polynomial (recall that $m_i(\mu) > 0$ for all $i \geq 0$), we have the following characterization for μ-polynomials:

Lemma 4.1.4 ([29]). *A real polynomial $\sum_{i=0}^n a_i z^i$, $a_i \geq 0$, is a μ-polynomial $f_{K;E}^{\mu}(z)$ for a measure μ on $\mathbb{R}_{\geq 0}$ and a pair $K, E \in \mathcal{K}^n$, with $\dim K = n-k$, $\dim E = r$, $\dim(K+E) = n$, if and only if $a_i > 0$ for all $k \leq i \leq r$, and $a_i = 0$ otherwise, and the sequence $\big(a_i/m_i(\mu)\big)_{i=0}^n$ is ultra-logconcave. Moreover, $\nu_1, \nu_2, \ldots, \nu_r \in \mathbb{C}$ are the roots of the μ-polynomial $f_{K;E}^{\mu}(z)$ of degree $r \leq n$, $\dim E = r$, $\dim K = n-k$, $\dim(K+E) = n$, if and only if*

$$\begin{aligned} \text{i)} \quad & (-1)^i \, \mathrm{s}_i\big(\nu_1, \ldots, \nu_r\big) > 0, \quad 0 \leq i \leq r-k, \\ & \mathrm{s}_i\big(\nu_1, \ldots, \nu_r\big) = 0, \quad r-k+1 \leq i \leq r, \\ \text{ii)} \quad & c_{r-i,n}^{\mu} \, \mathrm{s}_i\big(\nu_1, \ldots, \nu_r\big)^2 \geq \mathrm{s}_{i-1}\big(\nu_1, \ldots, \nu_r\big) \, \mathrm{s}_{i+1}\big(\nu_1, \ldots, \nu_r\big), \quad 1 \leq i \leq r-1. \end{aligned} \tag{4.8}$$

We notice that an analogous result to Lemma 4.1.4 can be stated for **m**-polynomials $f_{K;E}^{\mathbf{m}}(z)$.

Regarding the topology (closeness) of the cone $\mathcal{R}^{\mu}(n)$, we have the next result. Its proof follows similar steps to the corresponding one for Steiner polynomials (see Theorem 1.2 in [27]); we include it here for completeness.

Theorem 4.1.4 ([29]). *For any measure μ, the cone $\mathcal{R}^{\mu}(n)$ is closed.*

Proof. Let $\nu \in \operatorname{bd} \mathcal{R}^{\mu}(n)$. Since $\mathbb{R}_{\leq 0} \subset \mathcal{R}^{\mu}(n)$ (see Theorem 4.1.3), we may assume without loss of generality that $\nu \notin \mathbb{R}$. Let $(\nu_j)_{j \in \mathbb{N}} \subsetneq \operatorname{int} \mathcal{R}^{\mu}(n)$ be such that $\lim_{j \to \infty} \nu_j = \nu$. For each $j \in \mathbb{N}$, since $\nu_j \in \mathcal{R}^{\mu}(n)$, there exist $K_j, E_j \in \mathcal{K}^n$ with $\dim(K_j + E_j) = n$ such that $f_{K_j;E_j}^{\mu}(\nu_j) = 0$.

We notice that among all pairs of convex bodies with ν_j as a root of the corresponding μ-polynomial, we can always choose K_j, E_j such that $f_{K_j;E_j}^{\mu}(1) = 1$; otherwise, since $f_{K_j;E_j}^{\mu}(1) > 0$, we may consider the convex bodies

$$K_j' = \frac{1}{f_{K_j;E_j}^{\mu}(1)^{1/n}} K_j, \qquad E_j' = \frac{1}{f_{K_j;E_j}^{\mu}(1)^{1/n}} E_j,$$

for which we clearly have $f_{K_j';E_j'}^{\mu}(\nu_j) = 0$ and $f_{K_j';E_j'}^{\mu}(1) = 1$.

Since $f_{K_j;E_j}^{\mu}(1) = \sum_{i=0}^n \binom{n}{i} \mathrm{W}_i(K_j; E_j) m_i(\mu) = 1$, then $\mathrm{W}_i(K_j; E_j) \in \big[0, 1/\min_{0 \leq l \leq n}\{m_l(\mu)\}\big]$, for all $i = 0, \ldots, n$, and not all of them are zero. Denoting by $\mathrm{W}_{i,j} = \mathrm{W}_i(K_j; E_j)$, the bounded sequence of $(n+1)$-tuples of numbers $(\mathrm{W}_{0,j}, \ldots, \mathrm{W}_{n,j})_{j \in \mathbb{N}}$ has a convergent subsequence to an $(n+1)$-tuple $(\mathrm{W}_0, \ldots, \mathrm{W}_n)$, and without loss of generality we will assume that $(\mathrm{W}_{0,j}, \ldots, \mathrm{W}_{n,j})_{j \in \mathbb{N}}$ is the convergent subsequence.

On the one hand, by continuity, the numbers $\mathrm{W}_0, \ldots, \mathrm{W}_n$ also satisfy inequalities (1.20), and thus the sequence $\big\{a_i = \binom{n}{i} \mathrm{W}_i : i = 0, \ldots, n\big\}$ is ultra-logconcave. On the other hand,

$$\sum_{i=0}^n \binom{n}{i} \mathrm{W}_i \, m_i(\mu) = \lim_{j \to \infty} \sum_{i=0}^n \binom{n}{i} \mathrm{W}_{i,j} \, m_i(\mu) = \lim_{j \to \infty} f_{K_j;E_j}^{\mu}(1) = 1,$$

i.e., the polynomial

$$\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i m_i(\mu)z^i=\sum_{i=0}^{n}a_i m_i(\mu)z^i\neq 0.$$

Moreover, by continuity, the numbers $\mathrm{W}_0,\dots,\mathrm{W}_n$ also satisfy inequalities (1.21). Therefore the property $a_i>0$ for all $k\leq i\leq r$ and $a_i=0$ otherwise, holds for suitable $r,k\in\{0,\dots,n\}$. Then Lemma 4.1.4 ensures that $\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i m_i(\mu)z^i$ is a μ-polynomial of two convex bodies $K,E\in\mathcal{K}^n$ with $\dim K=n-k$, $\dim E=r$. By continuity, since $f^{\mu}_{K_j;E_j}(\nu_j)=0$ for all $j\in\mathbb{N}$ and the sequence of complex numbers $(\nu_j)_{j\in\mathbb{N}}$ converges to ν, we have $f^{\mu}_{K;E}(\nu)=0$, i.e., $\nu\in\mathcal{R}^{\mu}(n)$. This shows that the cone $\mathcal{R}^{\mu}(n)$ is closed. □

According to the above theorem, the 'geometry' of the set $\mathcal{R}^{\mu}(n)$ is given by the 'upper ray' of the boundary. Regarding the possible inclusion of this ray in the cone and the monotonicity in the dimension, we have the following results.

Proposition 4.1.2. *For any measure μ, the inclusion $\mathcal{R}^{\mu}(n)\subset\mathcal{R}^{\mu}(n+1)$ holds.*

Proof. Let $\nu\in\mathcal{R}^{\mu}(n)$ and let $K,E\in\mathcal{K}^n$, $\dim(K+E)=n$, be such that $f^{\mu}_{K;E}(\nu)=0$. Embedding K canonically into the hyperplane $\mathrm{e}_{n+1}^{\perp}\subsetneq\mathbb{R}^{n+1}$, let $K'=K\times[0,\mathrm{e}_{n+1}]$ be the prism over K of height 1 in the direction e_{n+1}. Then

$$\mathrm{vol}_{n+1}(K'+\lambda E)=\mathrm{vol}_n(K+\lambda E)$$

for all $\lambda\geq 0$, and thus (cf. (1.5))

$$\binom{n+1}{i}\mathrm{W}_i^{(n+1)}(K';E)=\binom{n}{i}\mathrm{W}_i^{(n)}(K;E),\quad i=0,\dots,n,$$
$$\mathrm{W}_{n+1}^{(n+1)}(K';E)=0.$$

Multiplying the above identities by $m_i(\mu)$ and $m_{n+1}(\mu)$ respectively, we obtain $f^{\mu}_{K';E}(z)=f^{\mu}_{K;E}(z)$, and thus $f^{\mu}_{K';E}(\nu)=0$. Hence $\nu\in\mathcal{R}^{\mu}(n+1)$. □

Next we show that the inclusion $\mathcal{R}^{\mu}(n)\subset\mathcal{R}^{\mu}(n+1)$ is strict. The proof follows similar steps to the one of Theorem 1.3 in [27]. We include it here for the sake of completeness.

Theorem 4.1.5 ([29]). *For any measure μ, $\mathcal{R}^{\mu}(n)$ is strictly increasing in the dimension, i.e., $\mathcal{R}^{\mu}(n)\subsetneq\mathcal{R}^{\mu}(n+1)$.*

Proof. In order to show that the above inclusion is indeed strict, let $\nu\in\mathrm{bd}\,\mathcal{R}^{\mu}(n)\setminus\mathbb{R}_{\leq 0}$; otherwise the assertion is trivially true (see e.g. Theorem 4.1.6 for $n=2$, and Theorem 1.2 in [26]). Since $\mathcal{R}^{\mu}(n)$ is closed (Theorem 4.1.4), ν is a root of some μ-polynomial $f^{\mu}_{K;E}(z)$ of degree $r\leq n$, with $K,E\in\mathcal{K}^n$, $\dim K=n-k$, $\dim E=r$, $\dim(K+E)=n$. Let $\overline{\nu},\nu_3,\dots,\nu_r$ be the remaining roots of the polynomial.

Then, we have to see that there exists $\varepsilon > 0$ small enough such that, for any $z \in \mathbb{C}$, $|z| = 1$, the r numbers $\nu + \varepsilon z, \overline{\nu} + \varepsilon\overline{z}, \nu_3, \dots, \nu_r$ are roots of a suitable μ-polynomial $f^{\mu}_{K';E'}(z)$, $K', E' \in \mathcal{K}^{n+1}$ with $\dim K' = n - k + 1$, $\dim E' = r$ and $\dim(K' + E') = n + 1$, i.e., the 'sign conditions' i) and the 'quadratic conditions' ii) in Lemma 4.1.4 are properly satisfied, namely,

$$\begin{aligned} \text{i')} \quad (-1)^i \,\mathrm{s}_i\big(\nu + \varepsilon z, \overline{\nu} + \varepsilon\overline{z}, \nu_3, \dots, \nu_r\big) &> 0, \quad 0 \le i \le r - k, \\ \mathrm{s}_i\big(\nu + \varepsilon z, \overline{\nu} + \varepsilon\overline{z}, \nu_3, \dots, \nu_r\big) &= 0, \quad r - k + 1 \le i \le r, \end{aligned}$$

and

$$\begin{aligned} \text{ii')} \quad & c^{\mu}_{r-i,n+1}\,\mathrm{s}_i\big(\nu + \varepsilon z, \overline{\nu} + \varepsilon\overline{z}, \nu_3, \dots, \nu_r\big)^2 \\ & \ge \mathrm{s}_{i-1}\big(\nu + \varepsilon z, \overline{\nu} + \varepsilon\overline{z}, \nu_3, \dots, \nu_r\big)\,\mathrm{s}_{i+1}\big(\nu + \varepsilon z, \overline{\nu} + \varepsilon\overline{z}, \nu_3, \dots, \nu_r\big), \end{aligned}$$

for $1 \le i \le r - 1$.

Since k of the r numbers $\nu + \varepsilon z, \overline{\nu} + \varepsilon\overline{z}, \nu_3, \dots, \nu_r$ are zero, we also have that, for any $\varepsilon > 0$,

$$\mathrm{s}_i\big(\nu + \varepsilon z, \overline{\nu} + \varepsilon\overline{z}, \nu_3, \dots, \nu_r\big) = 0 \quad \text{for } i \ge r - k + 1. \tag{4.9}$$

Obviously, the numbers $\nu + \varepsilon z, \overline{\nu} + \varepsilon\overline{z}, \nu_3, \dots, \nu_r$, are roots of a polynomial with real coefficients. Hence, in view of (4.8) i) and the continuity of polynomials, there exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon \le \varepsilon_1$

$$(-1)^i\,\mathrm{s}_i\big(\nu + \varepsilon z, \overline{\nu} + \varepsilon\overline{z}, \nu_3, \dots, \nu_r\big) > 0, \quad 0 \le i \le r - k.$$

So, with (4.9) both conditions in i') are satisfied for $\varepsilon \le \varepsilon_1$.

Relation (4.9) also implies that the inequalities in ii') are certainly satisfied for $r - k \le i \le r - 1$. So it remains to consider $1 \le i < r - k$. By (4.8) ii) we know that

$$c^{\mu}_{r-i,n}\,\mathrm{s}_i\big(\nu, \overline{\nu}, \dots, \nu_r\big)^2 \ge \mathrm{s}_{i-1}\big(\nu, \overline{\nu}, \dots, \nu_r\big)\mathrm{s}_{i+1}\big(\nu, \overline{\nu}, \dots, \nu_r\big),$$

and since

$$\begin{aligned} c^{\mu}_{r-i,n+1} &= \frac{\binom{n+1}{r-i-1}\binom{n+1}{r-i+1}}{\binom{n+1}{r-i}^2}\,\frac{m_{r-i-1}(\mu)m_{r-i+1}(\mu)}{m_{r-i}(\mu)^2} \\ &> \frac{\binom{n}{r-i-1}\binom{n}{r-i+1}}{\binom{n}{r-i}^2}\,\frac{m_{r-i-1}(\mu)m_{r-i+1}(\mu)}{m_{r-i}(\mu)^2} = c^{\mu}_{r-i,n} \end{aligned}$$

for all $1 \le i \le r - 1$ and $\mathrm{s}_i\big(\nu, \overline{\nu}, \dots, \nu_r\big)^2 > 0$ for $0 \le i \le r - k$ (cf. (4.8) i)), we get that

$$c^{\mu}_{r-i,n+1}\,\mathrm{s}_i\big(\nu, \overline{\nu}, \dots, \nu_r\big)^2 > \mathrm{s}_{i-1}\big(\nu, \overline{\nu}, \dots, \nu_r\big)\mathrm{s}_{i+1}\big(\nu, \overline{\nu}, \dots, \nu_r\big)$$

for all $1 \le i < r - k$. Hence, as before, by continuity of polynomials, there exists $\varepsilon_2 > 0$ such that

$$\begin{aligned} & c^{\mu}_{r-i,n+1}\,\mathrm{s}_i\big(\nu + \varepsilon z, \overline{\nu} + \varepsilon\overline{z}, \nu_3, \dots, \nu_r\big)^2 \\ & > \mathrm{s}_{i-1}\big(\nu + \varepsilon z, \overline{\nu} + \varepsilon\overline{z}, \nu_3, \dots, \nu_r\big)\mathrm{s}_{i+1}\big(\nu + \varepsilon z, \overline{\nu} + \varepsilon\overline{z}, \nu_3, \dots, \nu_r\big) \end{aligned}$$

for all $0 < \varepsilon \le \varepsilon_2$ and $1 \le i < r - k$. Thus we obtain ii') for $\varepsilon \le \varepsilon_2$, and the assertion follows with $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. $\square$

The following proposition is also analogous to the corresponding one for Steiner polynomials (see Corollary 1.1 in [27]).

Proposition 4.1.3 ([29]). *For $n \geq 3$ and a given measure μ on $\mathbb{R}_{\geq 0}$, let $K, E \in \mathcal{K}^n$ be such that the μ-polynomial $f^{\mu}_{K;E}(z)$ has a root lying on $\operatorname{bd} \mathcal{R}^{\mu}(n) \setminus \mathbb{R}_{\leq 0}$. Then K, E are extremal sets for at least one Aleksandrov-Fenchel inequality.*

Proof. For $\nu \in \operatorname{bd} \mathcal{R}^{\mu}(n) \setminus \mathbb{R}_{\leq 0}$, let $K, E \in \mathcal{K}^n$, with $\dim E = r$, be such that $f^{\mu}_{K;E}(\nu) = 0$, and let $\overline{\nu}, \nu_3, \ldots, \nu_r$ be the remaining roots of $f^{\mu}_{K;E}(z)$. If K, E are not extremal sets in any Aleksandrov-Fenchel inequality, i.e., if we have strict inequalities in (1.20), then $r \geq n-1$ and for all $1 \leq i \leq r-1$

$$c^{\mu}_{r-i,n} \mathrm{s}_i(\nu, \overline{\nu}, \nu_3, \ldots, \nu_r)^2 > \mathrm{s}_{i-1}(\nu, \overline{\nu}, \nu_3, \ldots, \nu_r) \mathrm{s}_{i+1}(\nu, \overline{\nu}, \nu_3, \ldots, \nu_r).$$

Again, by the continuity of the elementary symmetric functions, for $\varepsilon > 0$ small enough, the numbers $\nu + \varepsilon z, \overline{\nu} + \varepsilon \overline{z}, \nu_3, \ldots, \nu_n$ are roots of a polynomial with real coefficients, satisfying conditions i) and ii) in Lemma 4.1.4 for any $z \in \mathbb{C}$ with $|z| = 1$. Thus $\{\nu + \varepsilon z : |z| = 1\} \subsetneq \mathcal{R}^{\mu}(n)$, a contradiction. □

The cone of roots for a fixed gauge body

Let $E \in \mathcal{K}^n$ be a fixed gauge body. The proof of Theorem 4.1.3 also says that $\mathcal{R}^{\mu}(n; E)$ (cf. (4.5)) is a convex cone, containing the non-positive real axis $\mathbb{R}_{\leq 0}$. However, other properties of $\mathcal{R}^{\mu}(n)$ cannot be extended to $\mathcal{R}^{\mu}(n; E)$. For instance, $\mathcal{R}^{\mu}(n; E)$ is, in general, not closed (see e.g. Theorem 1.2 in [26]). Here we give a sufficient condition for $\nu \in \operatorname{bd} \mathcal{R}^{\mu}(n; E) \setminus \mathbb{R}_{\leq 0}$ to lie in $\mathcal{R}^{\mu}(n; E)$, involving the circumradius.

Proposition 4.1.4 ([29]). *Let $\nu \in \operatorname{bd} \mathcal{R}^{\mu}(n; E) \setminus \mathbb{R}_{\leq 0}$. Let $(\nu_j)_{j \in \mathbb{N}} \subsetneq \operatorname{int} \mathcal{R}^{\mu}(n; E)$ be a sequence with $\lim_{j \to \infty} \nu_j = \nu$ and, for each $j \in \mathbb{N}$, let $K_j \in \mathcal{K}^n$, $\dim(K_j + E) = n$, be such that $f^{\mu}_{K_j;E}(\nu_j) = 0$. If there exists a subsequence $(K_{j_m})_{m \in \mathbb{N}} \subset (K_j)_{j \in \mathbb{N}}$ with $\dim K_{j_m} = n - k$ for all $m \in \mathbb{N}$, such that*

$$\lim_{m \to \infty} \frac{\mathrm{W}_k(K_{j_m}; E)}{\mathrm{R}(K_{j_m})^{n-k}} \neq 0,$$

then $\nu \in \mathcal{R}^{\mu}(n; E)$.

Proof. Without loss of generality, we may suppose that $\dim K_j = n - k$ for all $j \in \mathbb{N}$ and thus $\mathrm{W}_k(K_j; E) \neq 0$.

For each $j \in \mathbb{N}$, let $\widetilde{K}_j = \mathrm{R}(K_j)^{-1} K_j$ and $\widetilde{\nu}_j = \mathrm{R}(K_j)^{-1} \nu_j$, which is a root of $f^{\mu}_{\widetilde{K}_j;E}(z)$. Since quermassintegrals are translation invariant (and thus the corresponding μ-polynomial), it is not restrictive to assume that the origin 0 is the circumcenter of all K_j, and so, of $\widetilde{K}_j$. Then, $\widetilde{K}_j \subset B_n$ for all $j \in \mathbb{N}$, and Blaschke's Selection Theorem (see Theorem 1.1.2) ensures the existence of a subsequence of $(\widetilde{K}_j)_{j\in\mathbb{N}}$ converging to a convex body $K \in \mathcal{K}^n$; without loss of generality we may assume that $\lim_{j\to\infty}\widetilde{K}_j = K$. Then, by the continuity of the quermassintegrals, each coefficient $\binom{n}{i}\mathrm{W}_i(\widetilde{K}_j;E)m_i(\mu)$ of the polynomial $f^{\mu}_{\widetilde{K}_j;E}(z)$ converges to the corresponding coefficient $\binom{n}{i}\mathrm{W}_i(K;E)m_i(\mu)$ of $f^{\mu}_{K;E}(z)$, and moreover,

$$\begin{aligned}\mathrm{W}_k(K;E) = \mathrm{W}_k\left(\lim_{j\to\infty}\widetilde{K}_j;E\right) &= \lim_{j\to\infty}\mathrm{W}_k\big(\widetilde{K}_j;E\big) = \lim_{j\to\infty}\mathrm{W}_k\left(\frac{1}{\mathrm{R}(K_j)}K_j;E\right)\\ &= \lim_{j\to\infty}\frac{\mathrm{W}_k(K_j;E)}{\mathrm{R}(K_j)^{n-k}} \neq 0.\end{aligned}$$

Hence, from the fact that the roots of a polynomial are continuous functions of the coefficients of the polynomial (see Theorem 1.5.1), and since $f^{\mu}_{\widetilde{K}_j;E}(z)$ has degree $r=\dim E$ for all $j\in\mathbb{N}$, we get that there exist $r-k$ sequences of numbers $(\nu_j^1)_{j\in\mathbb{N}},\dots,(\nu_j^{r-k})_{j\in\mathbb{N}}$ such that $\nu_j^1,\dots,\nu_j^{r-k}$ are the $r-k$ non-zero roots of $f^{\mu}_{\widetilde{K}_j;E}(z)$ for all $j\in\mathbb{N}$, and with $\nu^i=\lim_{j\to\infty}\nu_j^i$, $i=1,\dots,r-k$, being the non-zero roots of $f^{\mu}_{K;E}(z)$. Now, since $\widetilde{\nu}_j$ was a non-zero root of $f^{\mu}_{\widetilde{K}_j;E}(z)$ for each j, taking subsequences if necessary, we may assume that $(\widetilde{\nu}_j)_{j\in\mathbb{N}}$ converges to a root of $f^{\mu}_{K;E}(z)$, say ν^1. Then

$$\frac{\nu}{\nu^1} = \frac{\lim_{j\to\infty}\nu_j}{\lim_{j\to\infty}\widetilde{\nu}_j} = \lim_{j\to\infty}\frac{\nu_j}{\widetilde{\nu}_j} = \lim_{j\to\infty}\mathrm{R}(K_j)\in\mathbb{R}_{>0},$$

which implies that $f^{\mu}_{(\nu/\nu^1)K;E}(\nu)=0$, as required. □

4.1.3 The smallest and largest cones of roots of μ-polynomials

We will start this subsection by showing that there exists a relation between (the cone of) the roots of the Steiner polynomial (cf. (4.6)) and the Wills μ-polynomials. The key to prove this connection will be a well-known inequality involving the moments of any measure μ. First we will show it for the case of the (relative) Wills polynomial $f^{g}_{K;E}(z)$ via the Aleksandrov-Fenchel inequalities.

Indeed, the Aleksandrov-Fenchel inequalities (1.20) for $\mathrm{W}_i(C_n)=\kappa_i$ and any value of the dimension yield the inequalities

$$m_i(g)^2 \le m_{i+1}(g)m_{i-1}(g), \quad i=1,2,\dots, \tag{4.10}$$

for the moments $m_i(g)=1/\kappa_i$ of the measure g associated to $G(t)=e^{-\pi t^2}$.

Moreover, if we had equality in one of the above inequalities, i.e., if for some index $i\ge 1$ it is $\mathrm{W}_i(C_n)^2=\mathrm{W}_{i-1}(C_n)\mathrm{W}_{i+1}(C_n)$, then the known equality case in Aleksandrov-Fenchel inequality for 0-symmetric convex bodies (see Theorem 1.4.4) would lead to a contradiction: it would imply that C_n is an $(n-i-1)$-tangential body of a ball with $i\neq 0$, which is not true. Thus, inequalities (4.10) are strict.

This fact can be extended to the moments of (almost) any measure; it will be shown and used in the proof of the following result.

Theorem 4.1.6 ([29]). *For any measure μ, the inclusion $\mathcal{R}(n) \subset \mathcal{R}^{\mu}(n)$ holds. Moreover, if μ satisfies $\mu\bigl(\mathbb{R}_{\geq 0} \setminus \{0, m_i(\mu)/m_{i-1}(\mu)\}\bigr) \neq 0$ for $i = 1, 2, \dots$, then the inclusion is strict.*

Proof. Let $\gamma \in \mathcal{R}(n)$. Then there exist convex bodies $K, E \in \mathcal{K}^n$ such that

$$f_{K;E}(\gamma) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)\gamma^i = 0.$$

Moreover, for any measure μ on $\mathbb{R}_{\geq 0}$, the Cauchy-Schwarz inequality (1.29) yields

$$\begin{aligned} m_i(\mu)^2 &= \left(\int_0^{\infty} t^i \,\mathrm{d}\mu(t)\right)^2 = \left(\int_0^{\infty} t^{\frac{i+1}{2}} t^{\frac{i-1}{2}} \,\mathrm{d}\mu(t)\right)^2 \\ &\leq \int_0^{\infty} t^{i+1} \,\mathrm{d}\mu(t) \int_0^{\infty} t^{i-1} \,\mathrm{d}\mu(t) = m_{i+1}(\mu) m_{i-1}(\mu), \end{aligned} \tag{4.11}$$

i.e., $1/m_i(\mu)$, $i = 0, \dots, n$, satisfy the Aleksandrov-Fenchel inequalities (1.20). Then, we get that the sequence of positive numbers $\binom{n}{i}\mathrm{W}_i(K;E)/m_i(\mu)$ is ultra-logconcave, and Lemma 4.1.4 ensures that the Steiner polynomial $f_{K;E}(z)$ is also a μ-polynomial for some convex bodies $K', E' \in \mathcal{K}^n$. Therefore, $\gamma \in \mathcal{R}^{\mu}(n)$, as required.

In order to prove the last assertion, we notice that equality in the Cauchy-Schwarz inequality (4.11) holds if and only if

$$t^{\frac{i+1}{2}} - \frac{m_i(\mu)}{m_{i-1}(\mu)} t^{\frac{i-1}{2}} \equiv 0$$

almost everywhere (see Theorem 1.4.10) or, in other words, if and only if

$$\mu\Bigl(\mathbb{R}_{\geq 0} \setminus \bigl\{0,\, m_i(\mu)/m_{i-1}(\mu)\bigr\}\Bigr) = 0, \quad i \geq 1.$$

Therefore, if a measure μ satisfies the condition of the theorem, inequalities (1.20) strictly hold for the values $1/m_i(\mu)$, $i = 0, \dots, n$, which implies that $c^{\mu}_{r-i,n} > c_{r-i,n}$; here, $r = \dim E = \dim E'$, as usual. Hence, denoting by $\gamma_2, \dots, \gamma_r$ the remaining roots of $f_{K;E}(z) = f^{\mu}_{K';E'}(z)$, we get

$$\begin{aligned} c^{\mu}_{r-i,n} \mathrm{s}_i(\gamma, \gamma_2, \dots, \gamma_r)^2 &> c_{r-i,n} \mathrm{s}_i(\gamma, \gamma_2, \dots, \gamma_r)^2 \\ &\geq \mathrm{s}_{i-1}(\gamma, \gamma_2, \dots, \gamma_r) \mathrm{s}_{i+1}(\gamma, \gamma_2, \dots, \gamma_r), \end{aligned}$$

which implies, if $n \geq 3$, that $\gamma \in \operatorname{int} \mathcal{R}^{\mu}(n)$ (see the proof of Proposition 4.1.3). Therefore, $\mathcal{R}(n) \subsetneq \mathcal{R}^{\mu}(n)$ when $n \geq 3$. For $n = 2$, we just notice that the discriminant of $f^{\mu}_{K;E}(z)$ is

$$\Delta = m_1(\mu)^2 \left(4\mathrm{W}_1(K;E)^2 - 4\mathrm{vol}(K)\mathrm{vol}(E)\frac{m_0(\mu)m_2(\mu)}{m_1(\mu)^2}\right)$$

and thus, if $K = E = B_2$ we have $\Delta < 0$. Hence $\mathcal{R}(2) = \mathbb{R}_{\leq 0} \subsetneq \mathcal{R}^{\mu}(2)$. □

This result shows that the 'smallest' cone of roots of Wills μ-polynomials is the one given by the Steiner polynomial (see Figure 4.3). Furthermore, the known results for the roots of the Steiner polynomial (namely, for $n \geq 10$ there are Steiner polynomials having roots with strictly positive real part, see Proposition 1.3 in [27], and when n tends to ∞, $\mathcal{R}(n)$ covers the whole $\mathbb{C}^+$ except $\mathbb{R}_{>0}$, see Theorem 1.4 in [27]), together with the above result (cf. also Theorem 4.1.1) provide additional information about the roots of μ-polynomials when the dimension is large enough. This is the statement of the following corollary.

Corollary 4.1.2 ([29]). *If $n \geq 10$ the inclusion $\{z \in \mathbb{C}^+ : \operatorname{Re} z \leq 0\} \subsetneq \mathcal{R}^\mu(n)$ holds. Moreover, given $\gamma \in \mathbb{C}^+ \setminus \mathbb{R}_{>0}$, there exists n_γ such that $\gamma \in \mathcal{R}^\mu(n)$ for all $n \geq n_\gamma$.*

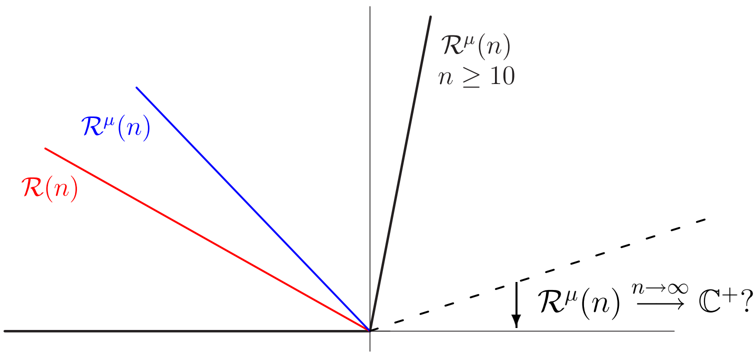

Figure 4.3: Relating $\mathcal{R}^\mu(n)$ and $\mathcal{R}(n)$.

Remark 4.1. *It is known that, for any $-a \in \mathbb{R}_{<0}$, $a > 0$, there exist convex bodies $K, E \in \mathcal{K}^n$ such that $-a$ is an n-fold root of the Steiner polynomial $f_{K;E}(z)$ (see e.g. Proposition 2.3 in [27]). This property remains true for any μ-polynomial. Indeed, since $1/m_i(\mu)$, $i = 0, \dots, n$, satisfy the Aleksandrov-Fenchel inequalities (1.20) (see (4.11)), then Lemma 4.1.4 ensures that*

$$\sum_{i=0}^{n} \binom{n}{i} z^i = (z+1)^n = f^\mu_{K';E'}(z)$$

is a μ-polynomial for two convex bodies $K', E' \in \mathcal{K}^n$, i.e., -1 is an n-fold root of $f^\mu_{K';E'}(z)$; for the real number $-a$, it suffices to consider $f^\mu_{aK';E'}(z)$.

Several of the above properties for the cone of roots of μ-polynomials (convexity, closeness, monotonicity in the dimension...) remain true for general $\mathbf{m}$-polynomials, independently the numbers m_i are moments of a measure on $\mathbb{R}_{\geq 0}$ or not; in fact, we have only needed that $m_i(\mu) > 0$ for all $i \geq 0$. However, the properties collected in Theorem 4.1.6 and Corollary 4.1.2 are not true for general/arbitrary $\mathbf{m}$-polynomials, as the following example shows.

Remark 4.2. *Let $\mathbf{m} = \left(e^{-i(i+1)/2}\right)_{i\in\mathbb{N}}$ and for any $K, E \in \mathcal{K}^n$ we consider the $\mathbf{m}$-polynomial $f^{\mathbf{m}}_{K;E}(z)$. It is easy to check that*

$$\frac{m_{i-1}}{m_i} \leq \beta \frac{m_{i+1}}{m_{i+2}} \quad \textit{for all} \;\; i = 1, 2, \dots, \tag{4.12}$$

where $\beta \approx 0.4655$ is the only real solution of the equation $z(z+1)^2 = 1$. Then, using (1.20) *and* (4.12) *we get that $f^{\mathbf{m}}_{K;E}(z)$ fulfills the stability criterion given by Theorem 1.5.2 for any pair of convex bodies $K, E \in \mathcal{K}^n$ and any value of the dimension. Therefore, Corollary 4.1.2 does not hold.*

Thus, Theorem 4.1.6 and Corollary 4.1.2 provide necessary 'geometric' conditions for a sequence of positive numbers $\{m_i : i = 0, 1, \dots\}$ to be the moments of a measure on $\mathbb{R}_{\geq 0}$. The question of knowing whether a sequence of positive numbers arises as the moments of a measure on $\mathbb{R}_{\geq 0}$ is a problem known in the literature as the *(Stieltjes) moment problem*, see e.g. [35]; nowadays there are still many open problems and on-going work on this topic.

So, we already know that the 'smallest' cone of roots of μ-polynomials is the one given by the Steiner polynomial. Next we deal with the 'largest' cone of roots of μ-polynomials, i.e., we would like to determine the 'largest' cone of roots containing $\mathcal{R}^{\mu}(n)$ for any measure μ on $\mathbb{R}_{\geq 0}$ (or for any measure μ on $\mathbb{R}_{\geq 0}$ belonging to a certain relevant subclass of measures).

We observe that, in the proof of Theorem 4.1.6, the main tool in order to get the desired inclusion was the inequality $m_{i+1}(\mu)m_{i-1}(\mu) \geq m_i(\mu)^2$ for all $i \geq 1$. So, for a 'reverse' inclusion we would need that

$$m_{i+1}(\mu)m_{i-1}(\mu) \leq c_i\, m_i(\mu)^2, \quad i \geq 1,$$

for a suitable sequence $(c_i)_{i\in\mathbb{N}}$. Theorem 4.1.7 determines such a sequence (and thus the corresponding inclusion) when working with *log-concave measures*, i.e., measures μ of the form

$$\mu(A) = \int_A f(t)\,\mathrm{d}t,$$

where f is a log-concave function (see Definition 1.6).

According to our terminology, and for any $\mathbf{m}$-polynomial, we will denote the corresponding cone of roots by $\mathcal{R}^{\mathbf{m}}(n)$, i.e.,

$$\mathcal{R}^{\mathbf{m}}(n) = \bigl\{z \in \mathbb{C}^+ : f^{\mathbf{m}}_{K;E}(z) = 0 \text{ for } K, E \in \mathcal{K}^n,\ \dim(K+E) = n\bigr\}.$$

Theorem 4.1.7 ([29]). *Let $\boldsymbol{\omega} = (i^i)_{i\in\mathbb{N}} = (1, 1, 2^2, \dots)$. Then $\mathcal{R}^{\mu}(n) \subset \mathcal{R}^{\boldsymbol{\omega}}(n)$ for any log-concave measure μ on $\mathbb{R}_{\geq 0}$.*

Proof. We take the real functions $f, g : [0,\infty) \longrightarrow [0,\infty)$ given by $f(t) = t^{i-1}$, $g(t) = t^{i+1}$, and let

$$h(t) = \sup\left\{f(x)^{1/2}g(y)^{1/2} : t = \frac{x+y}{2},\ x, y \in [0,\infty)\right\}.$$

Then, we clearly have $h\bigl((x+y)/2\bigr) \geq f(x)^{1/2}g(y)^{1/2}$ for all $x, y \in [0,\infty)$ and hence (since μ is log-concave), by the Prékopa-Leindler inequality (1.27),

$$\int_0^\infty h(t)\,\mathrm{d}\mu(t) \geq \left(\int_0^\infty t^{i-1}\mathrm{d}\mu(t)\right)^{1/2} \left(\int_0^\infty t^{i+1}\mathrm{d}\mu(t)\right)^{1/2},$$

i.e., we get

$$\left(\int_0^\infty h(t)\,\mathrm{d}\mu(t)\right)^2 \geq m_{i-1}(\mu)m_{i+1}(\mu). \tag{4.13}$$

So, now we deal with the left-hand side in the above inequality. Taking into account the definition of h, we maximize, for each $t \geq 0$, the function $F_t : [-t,t] \longrightarrow [0,\infty)$ given by

$$F_t(s) = (t-s)^{(i-1)/2}(t+s)^{(i+1)/2}.$$

Its first derivative $(F_t)'(s) = (t^2-s^2)^{(i-1)/2-1}\big(-is^2+(1-i)ts+t^2\big)$ has only one root in $(-t,t)$, namely, t/i, which gives in fact the maximum of the function. Therefore, we have

$$h(t) = F_t\left(\frac{t}{i}\right) = \begin{cases} t^i\dfrac{(i-1)^{(i-1)/2}(i+1)^{(i+1)/2}}{i^i} & \text{for } i>1,\\ 2t & \text{for } i=1. \end{cases}$$

So, denoting by

$$c_i = \frac{(i-1)^{(i-1)}(i+1)^{(i+1)}}{i^{2i}} \quad \text{for all } \ i>1, \quad c_1 = 4,$$

we get that

$$\left(\int_0^\infty h(t)\,\mathrm{d}\mu(t)\right)^2 = \left(\int_0^\infty c_i^{1/2}t^i\,\mathrm{d}\mu(t)\right)^2 = c_i m_i(\mu)^2,$$

and thus (4.13) yields

$$c_i m_i(\mu)^2 \geq m_{i-1}(\mu)m_{i+1}(\mu). \tag{4.14}$$

Now we can prove the result. Let $\nu \in \mathcal{R}^\mu(n)$. Then there exist convex bodies $K,E \in \mathcal{K}^n$ such that $f^\mu_{K;E}(\nu) = \sum_{i=0}^n \binom{n}{i}\mathrm{W}_i(K;E)\,m_i(\mu)\nu^i = 0$.

Let $\boldsymbol{\omega} = (\omega_i)_{i\in\mathbb{N}} = (1,1,2^2,\dots,i^i,\dots)$. By (1.20) and (4.14), we easily get that the sequence of positive numbers $\binom{n}{i}\mathrm{W}_i(K;E)m_i(\mu)/\omega_i$, $i=0,\dots,n$, is ultra-logconcave, and hence, $f^\mu_{K;E}(z)$ is also an $\boldsymbol{\omega}$-polynomial $f^{\boldsymbol{\omega}}_{K';E'}(z)$ for some convex bodies $K',E' \in \mathcal{K}^n$ (cf. Lemma 4.1.4). Thus, $f^{\boldsymbol{\omega}}_{K';E'}(\nu) = 0$, i.e., $\nu \in \mathcal{R}^{\boldsymbol{\omega}}(n)$, as required. □

Appendix: The 'largest cone' of roots is also the cone of a certain μ-polynomial

In the following, we deal with a 'moment problem': the question whether the 'weights' $m_i = i^i$ are the moments of a measure on $[0,\infty)$, i.e., whether $f^{\boldsymbol{\omega}}_{K;E}(z)$ is a Wills μ-polynomial.

Assuming that there exists such a measure μ, it must be a probability measure on $[0,\infty)$ because $m_0(\mu) = 1$. Then the moment generating function of such probability distribution is given by

$$\int_0^\infty e^{tx}\,\mathrm{d}\mu(x) = \sum_{k=0}^\infty \frac{m_k(\mu)}{k!}\,t^k = \sum_{k=0}^\infty \frac{k^k}{k!}\,t^k,$$

where, as usual, the above series is computed on its convergence disc.

Now, we consider the *Lambert function* W (also called the *omega function* or *product logarithm*) given by the inverse relation

$$z = W(z)e^{W(z)}, \tag{4.15}$$

that is, W is the inverse (multi-valued) function of $z \mapsto ze^z$. We observe that since $z \mapsto ze^z$ is not injective, W is, in general, multi-valued. Restricting our attention to real-valued W's, and adding the extra constraint $W \geq -1$, (4.15) allows to choose a single-valued function $W_0(x)$ with $W_0(0) = 0$ and $W_0(-1/e) = -1$. Such function W_0 is called the *main branch* of the multi-valued W (see [13] for further information about this topic). Using the Lagrange inversion theorem (see e.g. [1]), it can be shown that its Taylor series (around 0) is

$$W_0(z) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} z^k.$$

Although the radius of convergence of the above series is $1/e$, the function defined by this series can be extended to a holomorphic function on $\mathbb{C}$ with a branch cut along the interval $(-\infty, -1/e]$; this holomorphic function defines the *main branch* of the Lambert function W.

Thus, considering $F(s) = 1 - sW_0'(s)$, we have

$$F(s) = 1 - sW_0'(s) = \sum_{k=0}^{\infty} \frac{k^k}{k!} (-s)^k = \int_0^{\infty} e^{-sx} \,\mathrm{d}\mu(x) = \mathcal{L}(\mu)(s),$$

where $\mathcal{L}(\mu)$ denotes the Laplace transform of μ (see e.g. [14]). Notice also that F is an analytic function on $\mathrm{Re}(z) > -1/e$; therefore the measure μ given by

$$\mu(A) = \int_A f(x) \,\mathrm{d}x,$$

where $f(x) = \mathcal{L}^{-1}(F)(x)$ is the inverse Laplace transform of F, is the desired measure on $[0, \infty)$.

4.1.4 On the stability of generalized Wills μ-polynomials

We conclude the section considering the stability of Wills μ-polynomials, i.e., we study the inclusion

$$\mathcal{R}^{\mu}(n) \subset \big\{z \in \mathbb{C}^+ : \mathrm{Re}\, z < 0\big\} \cup \{0\},$$

property that we call, following the notation in [27], *'weak' stability.*

Of course, it is not possible to state a general characterization result for μ-polynomials for any measure μ. So, it is natural to consider particularly prominent polynomials of this type, which, at the end, will provide information on the stability of any μ-polynomial.

Theorem 4.1.6 shows that the 'smallest' cone of roots of μ-polynomials is the one given by the Steiner polynomial, and it is known that Steiner polynomials are weakly stable if and only if $n \leq 9$ (see Proposition 1.3 in [27]). Therefore, we can state the following result (see Corollary 4.1.2):

Corollary 4.1.3 ([29]). *If $n \geq 10$ then, for any measure μ on $\mathbb{R}_{\geq 0}$, μ-polynomials are not weakly stable, i.e.,* $\left\{z \in \mathbb{C}^+ : \operatorname{Re} z \leq 0\right\} \subsetneq \mathcal{R}^\mu(n)$.

Thus, we wonder for the stability of those polynomials which determine the 'largest' cone of roots containing $\mathcal{R}^\mu(n)$ for any log-concave measure μ on $\mathbb{R}_{\geq 0}$, i.e., the $\boldsymbol{\omega}$-polynomials (Theorem 4.1.7). We prove the following result.

Proposition 4.1.5 ([29]). *$\boldsymbol{\omega}$-polynomials are weakly stable if and only if $n \leq 3$.*

Proof. First we notice that the stability criterion given by Theorem 1.5.2 cannot be applied to $f^{\boldsymbol{\omega}}_{K;E}(z)$ when $n = 3$ because condition (1.31) is, in general, not fulfilled. Thus, we assume that $\boldsymbol{\omega}$-polynomials are not weakly stable in dimension $n = 3$, and hence, since the cone of roots is convex (cf. Theorem 4.1.3), we know that there exist $K, E \in \mathcal{K}^3$ such that $\mathrm{i}, -\mathrm{i}, -c$, $c \geq 0$, are the three roots of $f^{\boldsymbol{\omega}}_{K;E}(z)$. Then, it is an easy computation to check that the Aleksandrov-Fenchel inequalities (1.20) yield, in terms of c, the relations $4 \geq 3c^2 \geq 16/3$, a contradiction. Therefore, $\boldsymbol{\omega}$-polynomials are weakly stable for $n = 3$ and, from the monotonicity of the cone of the roots (see Theorem 4.1.5), also for $n = 2$.

Finally, we consider the $\boldsymbol{\omega}$-polynomial $f^{\boldsymbol{\omega}}_{B_4;B_4}(z) = \kappa_4 \sum_{i=0}^4 \binom{4}{i} i^i z^i$. It can be checked with a computer or by applying the Routh-Hurwitz criterion (see Theorem 1.5.3) that $f^{\boldsymbol{\omega}}_{B_4;B_4}(z)$ has a root with positive real part ($\nu \approx 0.03838 + 0.20807\,\mathrm{i}$). The non-stability property for all $n \geq 4$ is deduced again from the monotonicity of the cones (Theorem 4.1.5). $\square$

Thus, using Theorem 4.1.7, the following result is a direct consequence of the above proposition.

Corollary 4.1.4 ([29]). *If $n \leq 3$ then, for any log-concave measure μ on $\mathbb{R}_{\geq 0}$, μ-polynomials are weakly stable.*

We remark that this bound for the dimension might be not best possible, since we do not know whether the measure that 'provides' the $\boldsymbol{\omega}$-polynomials is a *log-concave* measure on $\mathbb{R}_{\geq 0}$.

Another particular interesting μ-polynomial is the (relative) Wills polynomial $f^g_{K;E}(z)$. Its stability can be also characterized.

Proposition 4.1.6 ([29]). *The (relative) Wills polynomials $f^g_{K;E}(z)$ are weakly stable if and only if $n \leq 7$.*

Proof. It is easy to check, using (1.20), that the stability criterion given by Theorem 1.5.2 is fulfilled for $n = 7$. The weak stability property for all $n \leq 6$ follows from Theorem 4.1.5. Now we consider

$$f(z) = \sum_{i=1}^{6} \binom{8}{i} \frac{1}{\kappa_i} z^i,$$

which is a relative Wills polynomial $f^g_{K;E}(z)$ for some $K,E\in\mathcal{K}^8$ (see Lemma 4.1.4). It can be checked with a computer or by applying the Routh-Hurwitz criterion that $f(z)$ has a root with positive real part, namely, $\nu\approx 0.05244+0.94238\,\mathrm{i}$. The non-stability property for $n\geq 8$ is deduced again from Theorem 4.1.5. □

4.2 On properties for $\mathbf{m}$-polynomials of unit p-balls

In the previous section we have investigated the structure of the roots of the family of $\mathbf{m}$-polynomials of convex bodies when $\mathbf{m}$ is associated to a given measure μ on the non-negative real line $\mathbb{R}_{\geq 0}$. As it has been shown, such family of $\mathbf{m}$-polynomials arises from the natural generalization of the classical Wills functional/polynomial.

A particulary interesting case of $\mathbf{m}$-polynomial associated to a measure on $\mathbb{R}_{\geq 0}$ is the following one. Let $G_p(t)=e^{-C_pt^p}$, $1\leq p<\infty$, be the function associated to the measure (cf. (4.1))

$$\mu_p(A)=\int_A p\,C_p\,e^{-C_pt^p}t^{p-1}\,\mathrm{d}t$$

on the non-negative real line $\mathbb{R}_{\geq 0}$, where $C_p=\bigl(2\Gamma(1/p+1)\bigr)^p$. The interesting feature of this measure is that it can be checked that its moments are the inverse of the volumes of unit p-balls, i.e., $m_i(\mu_p)=1/\kappa_i^p$, $i=0,\dots,n$, and thus it provides the natural extension of the classical Wills polynomial (with respect to the Euclidean unit ball) to the unit p-ball B_n^p.

Lemma 4.2.1 ([63]). *Let μ_p be the measure on the non-negative real line $\mathbb{R}_{\geq 0}$ associated to the function $G_p(t)=e^{-C_pt^p}$, $t\geq 0$, with $C_p=\bigl(2\Gamma(1/p+1)\bigr)^p$, $1\leq p<\infty$. Then the moments*

$$m_i(\mu_p)=\frac{1}{\kappa_i^p},\quad i\geq 0.$$

Proof. It is just an easy computation to check that (see (1.14))

$$m_i(\mu_p)=p\,C_p\int_0^\infty t^i\,e^{-C_pt^p}t^{p-1}\,\mathrm{d}t=\frac{1}{C_p^{i/p}}\int_0^\infty s^{i/p}e^{-s}\,\mathrm{d}s=\frac{\Gamma\left(\frac{i}{p}+1\right)}{\left(2\Gamma\left(\frac{1}{p}+1\right)\right)^i}=\frac{1}{\kappa_i^p}.$$ □

Remark 4.3. *If $p=\infty$, the corresponding measure μ_∞ is the discrete one given by $\mu_\infty\bigl(\{1/2\}\bigr)=1$, $\mu_\infty\bigl(\mathbb{R}_{\geq 0}\setminus\{1/2\}\bigr)=0$, for which $m_i(\mu_\infty)=1/\kappa_i^\infty$.*

Therefore, the μ_p-polynomials are given by

$$f^{\mu_p}_{K;E}(z)=\sum_{i=0}^n\binom{n}{i}\frac{\mathrm{W}_i(K;E)}{\kappa_i^p}z^i,\quad K,E\in\mathcal{K}^n.$$

In this section, we are mainly interested in studying several properties of the μ_p-polynomial $f^{\mu_p}_{K;B_n^p}(z)$, $K\in\mathcal{K}^n$. First we will study the (asymptotic) relation between its roots and the roots of the Steiner polynomial.

4.2.1 (Asymptotically) relating the roots of Steiner and $\mathbf{m}$-polynomials

At the beginning of (the first section of) this chapter, we have seen how some $\mathbf{m}$-polynomials appear in a natural way by means of a certain generalization of the Wills functional. In this subsection, our main goal is to provide another reason why these polynomials might be of interest: roughly speaking, they also arise when dealing with the asymptotic behavior of the roots of the Steiner polynomial.

First, we will give a general asymptotic relation involving the roots of Steiner polynomials and $\mathbf{m}$-polynomials (Theorem 4.2.1), and then we will particularize it to provide the connection between the roots of the Steiner polynomial and the μ_p-polynomials $f^{\mu_p}_{K;B^p_n}(z)$ (Theorem 4.2.2).

Some preliminary results

Here we collect several results which will be needed in the proofs of the main theorems. The proof of the first lemma includes the construction of a special family of gauge bodies which will be used in the following.

Lemma 4.2.2 ([63]). *Let $[a,b]$ be a closed interval in $\mathbb{R}$ with $0 \in [a,b]$, and let $r:[a,b] \longrightarrow [0,\infty)$ be a continuous concave (and not identically zero) function. Then there exists a sequence of convex bodies $\{E_n\}_{n\in\mathbb{N}}$ with $\dim E_n = n$, such that*

$$\frac{\mathrm{vol}_n(E_n)}{\mathrm{vol}_{n-k}(E_{n-k})} = \prod_{i=0}^{k-1} \int_a^b r(t)^{n-k+i}\mathrm{d}t, \quad 2 \le k \le n.$$

Proof. We consider the family of convex bodies inductively defined by

$$E_0 = \{0\}, \quad E_1 = [a,b], \quad E_n = \bigcup_{t\in[a,b]} \Big(r(t)E_{n-1} \times \{t\}\Big), \quad n \ge 1. \tag{4.16}$$

From the concavity and the continuity of $r(t)$, it is easy to see that E_n is, in fact, a convex body in $\mathbb{R}^n$, and since $r(t)$ is not identically zero, $\dim E_n = n$. Moreover, we have that, for all $0 \le k \le n$,

$$\begin{aligned}\mathrm{vol}_n(E_n) &= \int_a^b \mathrm{vol}_{n-1}\big(r(t)E_{n-1}\big)\mathrm{d}t = \mathrm{vol}_{n-1}(E_{n-1}) \int_a^b r(t)^{n-1}\mathrm{d}t \\ &= \cdots = \mathrm{vol}_{n-k}(E_{n-k}) \prod_{i=0}^{k-1} \int_a^b r(t)^{n-k+i}\mathrm{d}t\end{aligned}$$

which gives the required identity. □

If for some fixed $s \in \mathbb{N}$, the limits

$$\lim_{n\to\infty} \frac{\left(\int_a^b r(t)^{n-1}\mathrm{d}t\right)^k}{\prod_{i=0}^{k-1} \int_a^b r(t)^{n-k+i}\mathrm{d}t}$$

exist and are positive, $k = 2, \ldots, s$, then we define (see Lemma 4.2.2)

$$\lambda_k = \lim_{n\to\infty} \frac{\left(\int_a^b r(t)^{n-1}\mathrm{d}t\right)^k}{\prod_{i=0}^{k-1}\int_a^b r(t)^{n-k+i}\mathrm{d}t} = \lim_{n\to\infty} \frac{\mathrm{vol}_{n-k}(E_{n-k})\,\mathrm{vol}_n(E_n)^k}{\mathrm{vol}_n(E_n)\,\mathrm{vol}_{n-1}(E_{n-1})^k} > 0, \tag{4.17}$$

$k = 2, \ldots, s$, and $\lambda_0 = \lambda_1 = 1$.

For $1 \le p < \infty$, we consider the function

$$r_p : [-1, 1] \longrightarrow [0, \infty) \quad \text{given by} \quad r_p(t) = \left(1 - |t|^p\right)^{1/p}. \tag{4.18}$$

We observe that the family of unit p-balls B_n^p (cf. (1.13)) can be derived from (4.16) using the function r_p. Next lemma shows that r_p satisfies the limit condition defining λ_k (cf. (4.17)).

Lemma 4.2.3 ([63]). *For all $k \ge 1$,*

$$\lim_{n\to\infty} \frac{\left(\int_{-1}^1 r_p(t)^{n-1}\mathrm{d}t\right)^k}{\prod_{i=0}^{k-1}\int_{-1}^1 r_p(t)^{n-k+i}\mathrm{d}t} = 1.$$

Proof. First we observe that, for any $i \ge 0$,

$$\begin{aligned}\int_{-1}^1 r_p(t)^i\,\mathrm{d}t &= 2\int_0^1 (1-t^p)^{i/p}\,\mathrm{d}t = \frac{4}{p}\int_0^{\pi/2} (\cos s)^{(2i/p)+1}(\sin s)^{(2/p)-1}\,\mathrm{d}s\\ &= \frac{2}{p}\,\mathrm{B}\left(\frac{i}{p}+1, \frac{1}{p}\right) = \frac{2i}{p(i+1)}\frac{\Gamma(\frac{i}{p})\Gamma(\frac{1}{p})}{\Gamma(\frac{i+1}{p})},\end{aligned}$$

where B denotes the beta function (see page 11). Then, it is an easy computation to check that, for all $k \ge 1$,

$$\frac{\left(\int_{-1}^1 r_p(t)^{n-1}\mathrm{d}t\right)^k}{\prod_{i=0}^{k-1}\int_{-1}^1 r_p(t)^{n-k+i}\mathrm{d}t} = \frac{\left(\frac{n-1}{n}\right)^k \left(\Gamma\left(\frac{n-1}{p}\right)/\Gamma\left(\frac{n}{p}\right)\right)^k}{\left(\frac{n-k}{n}\right)\left(\Gamma\left(\frac{n-k}{p}\right)/\Gamma\left(\frac{n}{p}\right)\right)},$$

and since

$$\lim_{n\to\infty}\frac{\left((n-1)/n\right)^k}{\left((n-k)/n\right)} = 1,$$

it suffices to prove that

$$\lim_{n\to\infty}\frac{\left(\Gamma\left(\frac{n-1}{p}\right)/\Gamma\left(\frac{n}{p}\right)\right)^k}{\Gamma\left(\frac{n-k}{p}\right)/\Gamma\left(\frac{n}{p}\right)} = 1. \tag{4.19}$$

Then, by means of Stirling's formula (1.18) for the gamma function, we get

$$\begin{aligned}\lim_{n\to\infty}\frac{\left(\Gamma\left(\frac{n-1}{p}\right)/\Gamma\left(\frac{n}{p}\right)\right)^k}{\Gamma\left(\frac{n-k}{p}\right)/\Gamma\left(\frac{n}{p}\right)} &= \lim_{n\to\infty}\left(\frac{\left(\frac{n-1}{e}\right)^{(n-1)/p}\frac{1}{\sqrt{n-1}}}{\left(\frac{n}{e}\right)^{\frac{n}{p}}\frac{1}{\sqrt{n}}}\right)^k \frac{\left(\frac{n}{e}\right)^{n/p}\frac{1}{\sqrt{n}}}{\left(\frac{n-k}{e}\right)^{\frac{n-k}{p}}\frac{1}{\sqrt{n-k}}}\\ &= \lim_{n\to\infty}\frac{(n-1)^{(n-1)k/p}\,n^{n/p}}{(n-k)^{(n-k)/p}\,n^{nk/p}} = 1.\end{aligned}$$

□

The main results

From the following result, a consequence for μ_p-polynomials involving the unit p-balls will be obtained.

Theorem 4.2.1 ([63]). *Let $s \in \mathbb{N}$ and $r : [a, b] \longrightarrow [0, \infty)$ be a continuous concave (non-zero) function, $0 \in [a, b]$, such that λ_k exists, $0 \le k \le s$ (cf. (4.17)). Let $K \in \mathcal{K}^s$ and $\mathbf{m} = \big(\lambda_{s-i}/\mathrm{vol}_i(E_i)\big)_{i\in\mathbb{N}}$, with $m_i = 0$ for $i > s$, and E_j defined by (4.16). Embedding $K \subsetneq \mathbb{R}^n$, $n > s$, let $\gamma_{1,n}, \dots, \gamma_{s,n}$ be the non-zero roots of $f_{K;E_n}(z)$ and let $\nu_1, \dots, \nu_s$ be the roots of the $\mathbf{m}$-polynomial $f^{\mathbf{m}}_{K;E_s}(z)$. Then, reordering if necessary,*

$$\lim_{n\to\infty} \frac{\mathrm{vol}_n(E_n)}{\mathrm{vol}_{n-1}(E_{n-1})}\gamma_{i,n} = \nu_i, \quad i = 1, \dots, s.$$

Proof. For $t \in \mathbb{R}$, let $H(t) = \big\{(x_1, \dots, x_n)^\intercal \in \mathbb{R}^n : x_n = t\big\}$. We may assume without loss of generality that $K \subsetneq H(0)$. Then,

$$\begin{aligned}
\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K; E_n)\lambda^i &= \mathrm{vol}(K + \lambda E_n)\\
&= \int_{\lambda a}^{\lambda b} \mathrm{vol}_{n-1}\big((K + \lambda E_n) \cap H(t)\big)\mathrm{d}t\\
&= \int_{\lambda a}^{\lambda b} \mathrm{vol}_{n-1}\Big(K + \lambda\, r\big(\tfrac{t}{\lambda}\big)\, E_{n-1}\Big)\mathrm{d}t\\
&= \sum_{i=0}^{n-1}\binom{n-1}{i}\mathrm{W}_i^{(n-1)}(K; E_{n-1}) \int_{\lambda a}^{\lambda b} \lambda^i\, r\big(\tfrac{t}{\lambda}\big)^i\, \mathrm{d}t\\
&= \sum_{i=0}^{n-1}\binom{n-1}{i}\mathrm{W}_i^{(n-1)}(K; E_{n-1}) \frac{1}{\mathrm{vol}_i(E_i)} \int_{\lambda a}^{\lambda b} \mathrm{vol}_i\Big(\lambda\, r\big(\tfrac{t}{\lambda}\big) E_i\Big)\mathrm{d}t\\
&= \sum_{i=0}^{n-1}\binom{n-1}{i}\mathrm{W}_i^{(n-1)}(K; E_{n-1}) \frac{\mathrm{vol}_{i+1}(E_{i+1})}{\mathrm{vol}_i(E_i)}\, \lambda^{i+1},
\end{aligned}$$

and identifying coefficients of both polynomials, we get that

$$\mathrm{W}_i(K; E_n) = \frac{i\,\mathrm{vol}_i(E_i)}{n\,\mathrm{vol}_{i-1}(E_{i-1})}\, \mathrm{W}_{i-1}^{(n-1)}(K; E_{n-1}).$$

Thus, using the above relation recursively, we obtain

$$\mathrm{W}_{n-s+j}(K; E_n) = \frac{\mathrm{vol}_{n-s+j}(E_{n-s+j})}{\binom{n}{n-s+j}} \frac{\binom{s}{j}\,\mathrm{W}_j^{(s)}(K; E_s)}{\mathrm{vol}_j(E_j)}, \quad j = 0, \dots, s,$$

and, since $\dim K = s$, the Steiner polynomial takes the form

$$\begin{aligned}
f_{K;E_n}(z) &= z^{n-s}\sum_{j=0}^{s}\binom{n}{n-s+j}\mathrm{W}_{n-s+j}(K; E_n)z^j\\
&= z^{n-s}\sum_{j=0}^{s}\mathrm{vol}_{n-s+j}(E_{n-s+j})\frac{\binom{s}{j}\,\mathrm{W}_j^{(s)}(K; E_s)}{\mathrm{vol}_j(E_j)}z^j.
\end{aligned}$$

Then, for all $i = 1, \dots, s$, $\gamma_{i,n}$ is a (non-zero) root of $f_{K;E_n}(z)$ if and only if the complex number $\tilde{\gamma}_{i,n} = \big(\mathrm{vol}_n(E_n)/\mathrm{vol}_{n-1}(E_{n-1})\big)\gamma_{i,n}$ satisfies

$$\sum_{j=0}^{s} \frac{\mathrm{vol}_{n-s+j}(E_{n-s+j})}{\mathrm{vol}_n(E_n)} \left(\frac{\mathrm{vol}_{n-1}(E_{n-1})}{\mathrm{vol}_n(E_n)}\right)^j \frac{\binom{s}{j}\,\mathrm{W}_j^{(s)}(K;E_s)}{\mathrm{vol}_j(E_j)}\,\tilde{\gamma}_{i,n}^j = 0,$$

or equivalently, dividing by $\big(\mathrm{vol}_{n-1}(E_{n-1})/\mathrm{vol}_n(E_n)\big)^s$, if and only if $\tilde{\gamma}_{i,n}$ is a root of the polynomial

$$\sum_{j=0}^{s} \frac{\mathrm{vol}_{n-(s-j)}\big(E_{n-(s-j)}\big)\mathrm{vol}_n(E_n)^{s-j}}{\mathrm{vol}_n(E_n)\mathrm{vol}_{n-1}(E_{n-1})^{s-j}} \frac{\binom{s}{j}\mathrm{W}_j^{(s)}(K;E_s)}{\mathrm{vol}_j(E_j)} z^j$$
$$= z^s + \frac{s\,\mathrm{W}_{s-1}^{(s)}(K;E_s)}{\mathrm{vol}_{s-1}(E_{s-1})} z^{s-1} + \sum_{j=0}^{s-2} \beta_{s-j,n} \frac{\binom{s}{j}\,\mathrm{W}_j^{(s)}(K;E_s)}{\mathrm{vol}_j(E_j)} z^j$$

where, for the sake of brevity we are writing, for each $k = 2, \dots, s$,

$$\beta_{k,n} = \frac{\mathrm{vol}_{n-k}\big(E_{n-k}\big)\mathrm{vol}_n(E_n)^k}{\mathrm{vol}_n(E_n)\mathrm{vol}_{n-1}(E_{n-1})^k}.$$

By assumption (cf. (4.17)), $\lim_{n\to\infty} \beta_{k,n} = \lambda_k$, $k = 2, \dots, s$, which shows that the pointwise limit

$$\lim_{n\to\infty} \left(z^s + \frac{s\,\mathrm{W}_{s-1}^{(s)}(K;E_s)}{\mathrm{vol}_{s-1}(E_{s-1})} z^{s-1} + \sum_{j=0}^{s-2} \beta_{s-j,n} \frac{\binom{s}{j}\,\mathrm{W}_j^{(s)}(K;E_s)}{\mathrm{vol}_j(E_j)} z^j \right) = f_{K;E_s}^{\mathbf{m}}(z).$$

This, together with the fact that the roots of a polynomial are continuous functions of the coefficients (see Theorem 1.5.1), concludes the proof. □

As a direct consequence of Theorem 4.2.1 for the unit p-balls we obtain Theorem 4.2.2. In a sense, this result is saying that for high dimension n, the (n-dimensional) Steiner polynomial $f_{K;B_n^p}(z) = \sum_{i=n-s}^{n} \binom{n}{i}\mathrm{W}_i(K;B_n^p)z^i$ of a convex body K with fixed dimension $\dim K = s$ 'behaves as' its μ_p-polynomial $f_{K;B_s^p}^{\mu_p}(z) = \sum_{i=0}^{s} \binom{s}{i}\mathrm{W}_i^{(s)}(K;B_s^p)/\kappa_i^p z^i$.

Theorem 4.2.2 ([63]). *For $s \in \mathbb{N}$ fixed, let $K \in \mathcal{K}^s$ and let $\nu_1, \dots, \nu_s$ be the roots of $f_{K;B_s^p}^{\mu_p}(z)$, $1 \le p \le \infty$. Embedding $K \subsetneq \mathbb{R}^n$, $n \ge s$, let $\gamma_{1,n}, \dots, \gamma_{s,n}$ be the non-zero roots of $f_{K;B_n^p}(z)$. Then, reordering if necessary,*

$$\lim_{n\to\infty} \frac{\kappa_n^p}{\kappa_{n-1}^p}\gamma_{i,n} = \nu_i, \quad i = 1, \dots, s.$$

Proof. For $1 \le p < \infty$, let r_p be the function given by (4.18), which yields the unit p-balls via (4.16), i.e., $E_i = B_i^p$.

Then, Lemma 4.2.3 ensures that $\lambda_k = 1$ for all $k \ge 1$ (cf. (4.17)), and thus we can apply Theorem 4.2.1 in order to get the required result. Notice that now, $m_i = (1/\kappa_i^p)$, $i = 1, \dots, s$.

Finally, we deal with $p=\infty$. In that case B_n^∞ is the n-dimensional regular cube with edge-length 2, and hence

$$\frac{\kappa_n^\infty}{\kappa_{n-1}^\infty}\gamma_{i,n}=\frac{2^n}{2^{n-1}}\gamma_{i,n}=2\gamma_{i,n}.$$

Now we observe that, since $\dim K=s$, identifying K with its canonical embedding in $\mathrm{e}_{s+1}^\perp\subsetneq\mathbb{R}^{s+1}$,

$$\begin{aligned}\sum_{i=1}^{s+1}\binom{s+1}{i}\mathrm{W}_i^{(s+1)}(K;B_{s+1}^\infty)\lambda^i&=\mathrm{vol}_{s+1}(K+\lambda B_{s+1}^\infty)=2\lambda\mathrm{vol}_s(K+\lambda B_s^\infty)\\&=2\lambda\sum_{i=0}^{s}\binom{s}{i}\mathrm{W}_i^{(s)}(K;B_s^\infty)\lambda^i,\end{aligned}$$

and identifying coefficients of both polynomials we get

$$\binom{s+1}{i}\mathrm{W}_i^{(s+1)}(K;B_{s+1}^\infty)=2\binom{s}{i-1}\mathrm{W}_{i-1}^{(s)}(K;B_s^\infty),\quad i=1,\dots,s+1.$$

Iterating this embedding-process till $K\subsetneq\mathbb{R}^n$, we finally get the identities

$$\binom{n}{i}\mathrm{W}_i(K;B_n^\infty)=2^{n-s}\binom{s}{i-(n-s)}\mathrm{W}_{i-(n-s)}^{(s)}(K;B_s^\infty),\quad i=n-s,\dots,n,$$

and hence,

$$\begin{aligned}f_{K;B_n^\infty}(z)&=\sum_{i=n-s}^{n}\binom{n}{i}\mathrm{W}_i(K;B_n^\infty)z^i=z^{n-s}\sum_{i=n-s}^{n}\binom{n}{i}\mathrm{W}_i(K;B_n^\infty)z^{i-n+s}\\&=z^{n-s}\sum_{i=n-s}^{n}2^{n-s}\binom{s}{i-(n-s)}\mathrm{W}_{i-(n-s)}^{(s)}(K;B_s^\infty)z^{i-n+s}\\&=(2z)^{n-s}\sum_{j=0}^{s}\binom{s}{j}\mathrm{W}_j^{(s)}(K;B_s^\infty)z^j\\&=(2z)^{n-s}\sum_{j=0}^{s}\binom{s}{j}\frac{\mathrm{W}_j^{(s)}(K;B_s^\infty)}{2^j}(2z)^j\\&=(2z)^{n-s}\sum_{j=0}^{s}\binom{s}{j}\frac{\mathrm{W}_j^{(s)}(K;B_s^\infty)}{\kappa_j^\infty}(2z)^j=(2z)^{n-s}f_{K;B_s^\infty}^{\mu_\infty}(2z).\end{aligned}$$

Therefore, $2\gamma_{i,n}=\nu_i$. It concludes the proof. □

4.2.2 On inequalities for m-polynomials

Here we will show that for the μ_p-polynomial $f_{K;B_n^p}^{\mu_p}(z)$, $K\in\mathcal{K}^n$, the corresponding functional in K obtained when $z=1$ can be bounded just by the last but one relative quermassintegral (Theorem 4.2.3). This property will be obtained as a consequence of a more general inequality for **m**-polynomials (Proposition 4.2.1).

Again, the inequalities (1.20) will be the main ingredient for the proof of this result. It generalizes (for suitable general '$\mathbf{m}$-functionals') the inequality obtained in [43] for the Wills functional (1.10), namely, that

$$W(K)=\sum_{i=0}^{n}\binom{n}{i}\frac{\mathrm{W}_i(K)}{\kappa_i}\leq e^{n\mathrm{W}_{n-1}(K)/\kappa_{n-1}}.$$

The proof follows the idea of the one in [43].

Proposition 4.2.1 ([63]). *Let* $\mathbf{m}=(m_i)_{i\in\mathbb{N}}$ *be a sequence of positive real numbers such that* $\big((n+1)m_n^2/(nm_{n-1}m_{n+1})\big)_{n\in\mathbb{N}}$ *is a decreasing sequence and with*

$$\lambda=\lim_{n\to\infty}\frac{(n+1)}{n}\frac{m_n^2}{m_{n-1}m_{n+1}}>0.$$

Then, denoting by

$$C_n(\lambda)=\begin{cases}\left(\frac{1}{\lambda}\right)^{n(n-1)/2} & \text{if } 0<\lambda<1,\\ 1 & \text{otherwise,}\end{cases}$$

the following inequality holds:

$$f^{\mathbf{m}}_{K;E}(1)\leq m_n\mathrm{vol}(E)\,C_n(\lambda)\,e^{n\,m_{n-1}\mathrm{W}_{n-1}(K;E)/(m_n\mathrm{vol}(E))}. \tag{4.20}$$

Proof. For the sake of brevity we will write $\widetilde{\mathrm{W}}_r=\binom{n}{n-r}\mathrm{W}_{n-r}(K;E)\,m_{n-r}$. Then, by means of the Aleksandrov-Fenchel inequalities (1.20) we get

$$\widetilde{\mathrm{W}}_r^2\geq\frac{r+1}{r}\frac{(n-r+1)\,m_{n-r}^2}{(n-r)\,m_{n-r-1}m_{n-r+1}}\widetilde{\mathrm{W}}_{r-1}\widetilde{\mathrm{W}}_{r+1},$$

and the monotonicity hypothesis yields $\widetilde{\mathrm{W}}_r^2\geq\big((r+1)/r\big)\lambda\,\widetilde{\mathrm{W}}_{r-1}\,\widetilde{\mathrm{W}}_{r+1}$. Thus

$$\frac{\widetilde{\mathrm{W}}_r}{\widetilde{\mathrm{W}}_{r+1}}\geq\frac{r+1}{r}\,\lambda\,\frac{\widetilde{\mathrm{W}}_{r-1}}{\widetilde{\mathrm{W}}_r}\geq\frac{r+1}{r-1}\,\lambda^2\,\frac{\widetilde{\mathrm{W}}_{r-2}}{\widetilde{\mathrm{W}}_{r-1}}\geq\cdots\geq\lambda^r\,(r+1)\frac{\widetilde{\mathrm{W}}_0}{\widetilde{\mathrm{W}}_1},$$

and consequently

$$\widetilde{\mathrm{W}}_r\leq\widetilde{\mathrm{W}}_0\,\frac{1}{\lambda^{r(r-1)/2}}\,\frac{1}{r!}\left(\frac{\widetilde{\mathrm{W}}_1}{\widetilde{\mathrm{W}}_0}\right)^r\leq\widetilde{\mathrm{W}}_0\,C_n(\lambda)\,\frac{1}{r!}\left(\frac{\widetilde{\mathrm{W}}_1}{\widetilde{\mathrm{W}}_0}\right)^r.$$

Therefore, summing in r, for $r=0,\dots,n$, we obtain

$$f^{\mathbf{m}}_{K;E}(1)\leq\widetilde{\mathrm{W}}_0\,C_n(\lambda)\,e^{\widetilde{\mathrm{W}}_1/\widetilde{\mathrm{W}}_0}.$$

□

Remark 4.4. *The sequence* $\mathbf{m}=(1)_{n\in\mathbb{N}}$ *trivially satisfies the conditions of Proposition 4.2.1 and hence, Steiner polynomials satisfy a* (4.20)*-type inequality, namely,*

$$f_{K;E}(1)\leq\mathrm{vol}(E)\,e^{n\mathrm{W}_{n-1}(K;E)/\mathrm{vol}(E)}.$$

Next we show that μ_p-functionals also satisfy a (4.20)-type inequality.

Theorem 4.2.3 ([63]). *Let $K \in \mathcal{K}^n$ and $1 \le p \le \infty$. Then*

$$\sum_{i=0}^{n} \binom{n}{i} \frac{\mathrm{W}_i(K; B_n^p)}{\kappa_i^p} \le e^{n\mathrm{W}_{n-1}(K;B_n^p)/\kappa_{n-1}^p}.$$

Proof. Since $f^{\mu_p}_{K;B_n^p}(z) = \sum_{i=0}^n \binom{n}{i}\mathrm{W}_i(K;B_n^p)/\kappa_i^p z^i$, we have to check that the conditions of Proposition 4.2.1 are satisfied for the sequence $(1/\kappa_n^p)_{n\in\mathbb{N}}$, $1 \le p \le \infty$.

First we notice that for $p = \infty$ we get

$$\frac{n+1}{n}\frac{\kappa_{n-1}^\infty \kappa_{n+1}^\infty}{(\kappa_n^\infty)^2} = \frac{n+1}{n}\frac{2^{n-1}2^{n+1}}{(2^n)^2} = \frac{n+1}{n},$$

which is clearly a decreasing sequence and $\lim_{n\to\infty}(n+1)/n = 1$. So, we assume $1 \le p < \infty$. On the one hand, it is easy to check that (cf. (1.14))

$$\frac{n+1}{n}\frac{\kappa_{n-1}^p \kappa_{n+1}^p}{(\kappa_n^p)^2} = \frac{n}{n-1}\frac{\Gamma\left(\frac{n}{p}\right)^2}{\Gamma\left(\frac{n-1}{p}\right)\Gamma\left(\frac{n+1}{p}\right)},$$

and using (4.19) for $k = 2$ we get that it converges to 1 when n goes to ∞. Therefore $\lambda = 1$ and so $C_n(\lambda) = 1$.

Thus, it remains to be studied the monotonicity of the above sequence, which, for convenience, can be also rewritten as

$$\frac{n+1}{n}\frac{\kappa_{n-1}^p \kappa_{n+1}^p}{(\kappa_n^p)^2} = \frac{\Gamma\left(\frac{n}{p}\right)\Gamma\left(\frac{n}{p}+1\right)}{\Gamma\left(\frac{n}{p}+1-\frac{1}{p}\right)\Gamma\left(\frac{n}{p}+\frac{1}{p}\right)}. \tag{4.21}$$

In order to do it, we take the real functions $f_1, f_2 : (0,\infty) \longrightarrow \mathbb{R}$, given by $f_1(x) = (x - 1/2)\log x$ and $f_2(x) = \eta(x)$ where $\eta(x)$ is the function defined by (2.4). The concavity of their first derivatives (see Lemma 2.1.1), f_i', $i = 1, 2$, together with the Mean-Value Theorem, allows to deduce that, in both cases $1 \le p \le 2$ and $p \ge 2$, we have

$$f_i'(x) + f_i'(x+1) - f_i'\left(x + \frac{1}{p}\right) - f_i'\left(x + 1 - \frac{1}{p}\right) < 0.$$

Hence, the real functions $h_i : (0,\infty) \longrightarrow \mathbb{R}$, $i = 1, 2$, given by

$$h_i(x) = f_i(x) + f_i(x+1) - f_i\left(x + \frac{1}{p}\right) - f_i\left(x + 1 - \frac{1}{p}\right),$$

are strictly decreasing, which implies that $e^{h_1(x)+h_2(x)}$ is so. Now, Stirling's formula (1.17) for the gamma function $\Gamma(x)$ allows to write

$$\frac{\Gamma(x)\Gamma(x+1)}{\Gamma(x+1-\frac{1}{p})\Gamma(x+\frac{1}{p})} = e^{h_1(x)+h_2(x)}.$$

Thus, all together, we can conclude that the sequence in (4.21) is strictly decreasing in n.

Therefore, all conditions in Proposition 4.2.1 are satisfied, and thus, inequality (4.20) for $E = B_n^p$ and $\mathbf{m} = (1/\kappa_n^p)_{n\in\mathbb{N}}$ shows that $f_{K;B_n^p}^{\mu_p}(1) \leq e^{n\mathrm{W}_{n-1}(K;B_n^p)/\kappa_{n-1}^p}$, as desired. □

4.3 On the roots of the Wills functional

In this section we deal with some (of the above studied) questions for the particular case of the (classical) Wills polynomial of convex bodies, as well as with the study of the size of its roots, bounding them in terms of functionals like the in- and circumradius of the set. We also relate the roots of the Steiner and the Wills polynomials.

To this end, and for the sake of brevity, in the following the *Wills polynomial* of $K \in \mathcal{K}^n$,

$$f_{K;B_n}^g(z) = \sum_{i=0}^{n} \binom{n}{i} \frac{\mathrm{W}_i(K)}{\kappa_i} z^i,$$

regarded as a formal polynomial in a complex variable $z \in \mathbb{C}$, will be (re)written as

$$g_K(z) = \sum_{i=0}^{n} \binom{n}{i} \frac{\mathrm{W}_{n-i}(K)}{\kappa_{n-i}} z^i = \sum_{i=0}^{n} \mathrm{V}_i(K) z^i.$$

Remark 4.5. *We would like to point out that, since*

$$g_K(z) = z^n f_{K;B_n}^g\left(\frac{1}{z}\right),$$

both real polynomials have essentially the 'same' non-zero roots, in the sense that if $\nu \in \mathbb{C}$, $\nu \neq 0$, is a root of $g_K(z)$, then $\nu/|\nu|^2$ is a root of $f_{K;B_n}^g(z)$ and vice versa. Moreover, by the homogeneity of the quermassintegrals we have that for any $K \in \mathcal{K}^n$ and all $\lambda > 0$, $g_{\lambda K}(z) = g_K(\lambda z)$; thus, for the sets of roots of such polynomials there will be no difference regarding their structure and description. In this way, the set $\mathcal{R}^g(n;B_n)$ (cf. (4.5)) turns out to be the set of all roots of $g_K(z)$, $K \in \mathcal{K}^n$, in the upper half-plane, plus the origin, *because $g_K(0) \neq 0$ for any $K \in \mathcal{K}^n$, since the constant term of $g_K(z)$ is always* 1.

We notice that $g_K(z)$ (and hence its roots) does not depend on the dimension of the space $\mathbb{R}^n$ where K is embedded, because the intrinsic volumes of K have this property. Thus, from now on and unless we explicitly say the opposite, we will always assume that for $K \in \mathcal{K}^n$, $\dim K = n$.

4.3.1 The cone of the roots of the Wills polynomial

We start this subsection studying the structure and the monotonicity of the set of roots of the Wills polynomial in the upper half-plane, i.e., $\mathcal{R}^g(n;B_n)$ (cf. (4.5)).

Theorem 4.3.1 ([30]). *$\mathcal{R}^g(n;B_n)$ is a convex cone, containing the non-positive real axis $\mathbb{R}_{\leq 0}$ and monotonous in the dimension, i.e., $\mathcal{R}^g(n;B_n) \subset \mathcal{R}^g(n+1;B_{n+1})$.*

Proof. The inclusion $\mathcal{R}^g(n;B_n) \subset \mathcal{R}^g(n+1;B_{n+1})$ is a direct consequence of the fact that intrinsic volumes remain unchanged if a convex body K is embedded in any Euclidean space of bigger dimension. The remaining properties are just a consequence of (the proof of) Theorem 4.1.3. □

Remark 4.6. *It is well-known (see e.g. Proposition 3 in [61]) that if P is an orthogonal box with edge-lengths $a_1,\dots,a_n > 0$, then the roots of $g_P(z)$ are $\nu_i = -1/a_i$, $i = 1,\dots,n$. In particular, the Wills polynomial of the n-dimensional cube of edge-length a, $g_{aC_n}(z)$, has an n-fold root $\nu = -1/a$.*

In the particular cases $n = 2, 3$, we can precisely describe the cones $\mathcal{R}^g(2;B_2)$ and $\mathcal{R}^g(3;B_3)$. Before giving this characterization we study the weak stability of the Wills polynomial, i.e., we study the inclusion

$$\mathcal{R}^g(n;B_n) \subset \bigl\{z \in \mathbb{C}^+ : \operatorname{Re} z < 0\bigr\} \cup \{0\},$$

since it will be needed in the proof of such result (Theorem 4.3.2). The main ingredient in order to do it will be again inequalities (1.20) and (1.21).

Proposition 4.3.1 ([30]). *Wills polynomials are weakly stable if $n \leq 7$. For $n \geq 14$ we have $\bigl\{z \in \mathbb{C}^+ : \operatorname{Re} z \leq 0\bigr\} \subsetneq \mathcal{R}^g(n;B_n)$.*

Proof. It is easy to check that (1.21) ensures that the stability criterion given by Theorem 1.5.2 is fulfilled for $n = 7$. The weak stability property for all $n \leq 6$ follows from the monotonicity of the cone of the roots (see Theorem 4.3.1). Finally, it can be checked with a computer or by applying the Routh-Hurwitz criterion (see Theorem 1.5.3) that the polynomial

$$g_{B_{14}}(z) = \kappa_{14} \sum_{i=0}^{14} \binom{14}{i} \frac{1}{\kappa_{14-i}} z^i$$

has a root with positive real part ($\nu \approx 0.04562 + 1.81036\mathrm{i}$). The non-stability property for all $n \geq 14$ is deduced again from the monotonicity of the cones (see Theorem 4.3.1). □

We observe that several of the above properties present restrictions in the dimension, in contrast to the known results for the roots of the relative Steiner polynomials ([27]) or, in a more general context, to those ones for the roots of the μ-polynomials, for any measure μ on $\mathbb{R}_{\geq 0}$ (see Section 4.1). It is due to the fact that in higher dimensions we do not have enough information about the so-called 'full system' of inequalities among the quermassintegrals (cf. e.g. Problem 6.1 in [21]).

So only in dimensions $8 \leq n \leq 13$ we do not know whether Wills polynomials may have roots with positive real parts. Obviously, by the convexity of the cone $\mathcal{R}^g(n;B_n)$, the existence of a root with positive real part implies the existence of a pure imaginary complex root. However, not all roots can be of that type. More precisely:

Proposition 4.3.2 ([30]). *There exists no convex body $K \in \mathcal{K}^n$ such that all roots of $g_K(z)$ are imaginary pure complex numbers (excluding the real root existing in odd dimension).*

The proof of this result follows similar steps to the one of the corresponding result for the Steiner polynomial (Proposition 2.1 in [26]). We include it here for completeness.

Proof. By Proposition 4.3.1 all roots of $g_K(z)$ are contained in the (open) left half-plane if $n \leq 7$, and so we may assume that $n = \dim K \geq 8$.

Let $K \in \mathcal{K}^n$ be a convex body, n even, such that all roots of $g_K(z)$ are $\{\pm b_j\,\mathrm{i}, j = 1, \dots, n/2\}$, with all $b_j > 0$. Then

$$g_K(z) = \sum_{i=0}^{n} \mathrm{V}_i(K) z^i = \mathrm{vol}(K) \prod_{j=1}^{n/2} (z^2 + b_j^2),$$

which implies $\mathrm{V}_{2i+1}(K) = 0$ for all $i = 0, \dots, (n-2)/2$. In particular, $\mathrm{V}_1(K) = 0$, i.e., $\dim K = 0$, a contradiction.

For n odd, let $K \in \mathcal{K}^n$ be such that the roots of $g_K(z)$ are $\{-a, \pm b_j\,\mathrm{i}, j = 1, \dots, (n-1)/2\}$, $a, b_j > 0$. Then

$$g_K(z) = \sum_{i=0}^{n} \mathrm{V}_i(K) z^i = \mathrm{vol}(K)(z + a) \prod_{j=1}^{(n-1)/2} (z^2 + b_j^2)$$

and, in particular, we have

$$\mathrm{vol}(K)\, a \prod_{j=1}^{(n-1)/2} b_j^2 = 1, \quad \mathrm{vol}(K) \prod_{j=1}^{(n-1)/2} b_j^2 = \mathrm{V}_1(K), \quad \mathrm{vol}(K)\, a = \mathrm{V}_{n-1}(K).$$

Thus we get the relation $\mathrm{V}_{n-1}(K)\mathrm{V}_1(K) = \mathrm{vol}(K)$, which implies, by (1.9) and inequality (1.21), that $\kappa_{n-1}/\kappa_n \geq n^2/2$. It contradicts the well-known inequality

$$\sqrt{\frac{2\pi}{n+1}} < \frac{\kappa_n}{\kappa_{n-1}} < \sqrt{\frac{2\pi}{n}} \tag{4.22}$$

(see e.g. Theorem 5.3.2 and page 216 in [58]) since $n > 1$. □

Next we come to the characterization of the cones $\mathcal{R}^g(2; B_2)$ and $\mathcal{R}^g(3; B_3)$.

Theorem 4.3.2 ([30]). *$\mathcal{R}^g(2; B_2) = \mathcal{R}^g(B_2; B_2)$ and $\mathcal{R}^g(3; B_3) = \mathcal{R}^g(B_3; B_3)$.*

Proof. We start determining the 2-dimensional cone $\mathcal{R}^g(2; B_2)$. Let $-a+b\mathrm{i} \in \mathbb{C}^+$ be a root of a Wills polynomial $g_K(z)$ for some planar convex body $K \in \mathcal{K}^2$. By Proposition 4.3.1 and Theorem 4.3.1 we may assume that both $a, b > 0$. Thus $g_K(z) = \mathrm{vol}(K)(z^2 + 2az + a^2 + b^2)$, and we have the identities $2\mathrm{vol}(K)\, a = \mathrm{V}_1(K)$, $\mathrm{vol}(K)(a^2 + b^2) = 1$, from which we get

$$\mathrm{vol}(K) = \frac{1}{a^2 + b^2}, \qquad \mathrm{V}_1(K) = \frac{2a}{a^2 + b^2}.$$

Then, the isoperimetric inequality (1.26) in terms of the intrinsic volumes (cf. (1.9)), namely,

$$\mathrm{V}_1(K)^2 \geq \pi \mathrm{vol}(K),$$

yields

$$b \leq \sqrt{\frac{4-\pi}{\pi}}\, a. \tag{4.23}$$

If we have equality in (4.23) then equality in the isoperimetric inequality holds, which implies that K is the Euclidean ball. Conversely, if $K = B_2$ then $g_{B_2}(z) = \pi z^2 + \pi z + 1$, whose (complex) roots give equality in (4.23). Therefore, equality holds in (4.23) if and only if $K = B_2$ (up to homotheties). This, together with the fact that $\mathcal{R}^g(2; B_2)$ is a cone (Theorem 4.3.1) shows that

$$\mathcal{R}^g(2; B_2) = \mathcal{R}^g(B_2; B_2) = \left\{ x + y\mathrm{i} \in \mathbb{C}^+ : \sqrt{\frac{4-\pi}{\pi}}\, x + y \leq 0 \right\}.$$

Now we consider the 3-dimensional case. Since $g_{B_3}(z) = (4\pi/3)z^3 + 2\pi z^2 + 4z + 1$, it can be checked that (cf. (4.3))

$$m_0 = \left| \tan \theta^g_{B_3;B_3} \right| = \frac{\sqrt{3}(t_- + t_+)}{t_- - t_+ + 2\sqrt{\pi}} \approx 0.9624,$$

where $t_\pm = \left(\sqrt{6\pi^2 - 39\pi + 64} \pm \sqrt{\pi}(\pi - 3)\right)^{1/3}$.

Let $-a + b\mathrm{i} \in \mathbb{C}^+$ be a root of a Wills polynomial $g_K(z)$ for some $K \in \mathcal{K}^3$. By Proposition 4.3.1 and Theorem 4.3.1 we may assume that both $a, b > 0$ and taking $m = b/a$, $m > 0$, we have to show that $m \leq m_0$. Let $-c$ be the real root of $g_K(z)$, $c > 0$. Then we have the identities

$$(2a + c) = \frac{\mathrm{V}_2(K)}{\mathrm{vol}(K)}, \quad (a^2 + b^2 + 2ac) = \frac{\mathrm{V}_1(K)}{\mathrm{vol}(K)}, \quad c(a^2 + b^2) = \frac{1}{\mathrm{vol}(K)}, \tag{4.24}$$

and using (1.9), inequalities (1.20) for $i = 1, 2$ yield, in terms of a, c, m,

$$\begin{aligned} &i) \quad \frac{4}{3}c^2 + \left(\frac{16}{3} - 2\pi\right) ac + \left[\frac{16}{3} - \pi(1 + m^2)\right] a^2 \geq 0, \\ &ii) \quad \left[4\pi - 8(1 + m^2)\right] c^2 + \left[4a(1 + m^2)(\pi - 4)\right] c + \pi a^2 (1 + m^2)^2 \geq 0, \end{aligned} \tag{4.25}$$

respectively.

We assume $m > m_0$. On the one hand it can be seen that, since $c > 0$, inequality (4.25) i) is equivalent to

$$c \geq \widetilde{c} = \frac{a\left(\sqrt{3\pi(4m^2 + 3\pi - 12)} + 3\pi - 8\right)}{4}.$$

On the other hand, a direct computation shows that the above condition on m also implies that inequality (4.25) ii) holds if and only if

$$0 < c \leq \bar{c} = \frac{a(m^2 + 1)\left(\sqrt{2(\pi m^2 - 3\pi + 8)} + \pi - 4\right)}{2(2m^2 - \pi + 2)}.$$

Hence, $\widetilde{c} \leq c \leq \bar{c}$, which is a contradiction because it can be checked that condition $m > m_0$ gives $\bar{c} < \widetilde{c}$. Therefore $m \leq m_0$, and using the convexity of the cone $\mathcal{R}^g(3; B_3)$ we get the result.

Moreover, since equality in (1.20) for $E = B_n$ and $i = 2$ holds only for the ball (see Theorem 1.4.3), an analogous argument to the one of the case $n = 2$ shows that the equality $m = m_0$ holds if and only if $K = B_3$ (up to homotheties). □

We observe that, in particular, $\mathcal{R}^g(2; B_2)$ and $\mathcal{R}^g(3; B_3)$ are closed convex cones, but we do not know whether this holds in general.

Remark 4.7. *From the above proof, it is also obtained that the ball B_n is the only convex body such that one of the roots of $g_{B_n}(z)$ lies on (determines) the boundary* $\operatorname{bd}\mathcal{R}^g(n; B_n)\backslash\mathbb{R}_{\leq 0}$*, $n = 2, 3$.*

Furthermore, from the proof of the above theorem, we may assert that these cones remain unchanged if any gauge body $E \in \mathcal{K}^n$ is considered, $n = 2, 3$. In other words:

Proposition 4.3.3 ([29]). *$\mathcal{R}^g(i) = \mathcal{R}^g(B_i; B_i)$, $i = 2, 3$.*

4.3.2 Relating the roots of the Wills and Steiner polynomials

In Theorem 1.2 of [26] and Proposition 1.2 in [27] it is proved that

$$\begin{aligned}
\mathcal{R}(2; B_2) &= \mathbb{R}_{\leq 0}, \\
\mathcal{R}(3; B_3) &= \left\{x + y\mathrm{i} \in \mathbb{C}^+ : x + \sqrt{3}\, y < 0\right\} \cup \{0\}, \\
\mathcal{R}(4; B_4) &= \left\{x + y\mathrm{i} \in \mathbb{C}^+ : x + y < 0\right\} \cup \{0\},
\end{aligned}$$

(cf. (4.7)). A first direct observation from Theorem 4.3.2 is that $\operatorname{cl}\mathcal{R}(n; B_n) \subsetneq \mathcal{R}^g(n; B_n)$ for dimensions $n = 2, 3$. Moreover, it is easy to check that in dimension 4, the cone

$$\mathcal{R}^g(B_4; B_4) = \left\{x + y\mathrm{i} \in \mathbb{C}^+ : \alpha x + y < 0\right\},$$

$\alpha = 1.42224\ldots$, and hence we also have the strict inclusion $\operatorname{cl}\mathcal{R}(4; B_4) \subsetneq \mathcal{R}^g(B_4; B_4) \subset \mathcal{R}^g(4; B_4)$. We cannot expect, however, that $\operatorname{cl}\mathcal{R}(n; B_n) \subsetneq \mathcal{R}^g(B_n; B_n)$ for any dimension; indeed, it can be checked with a computer or by applying the Routh-Hurwitz criterion that $g_{B_{12}}(z)$ is weakly stable, whereas the (weak) stability of the Steiner polynomial fails for $n = 12$ (see Remark 3.2 in [25]).

Let γ_i, $i = 1, \ldots, n$, be the roots of the Steiner polynomial $f_{K;B_n}(z)$ which can be rewritten as $f_{K;B_n}(z) = \sum_{i=0}^{n} \kappa_i \mathrm{V}_{n-i}(K) z^i$, for $K \in \mathcal{K}^n$. From the identity

$$\sum_{i=0}^{n} \kappa_i \mathrm{V}_{n-i}(K) z^i = \kappa_n \prod_{i=1}^{n} (z - \gamma_i)$$

we get

$$(-1)^i \frac{\kappa_{n-i}}{\kappa_n} \mathrm{V}_i(K) = \mathrm{s}_i\big(\gamma_1, \ldots, \gamma_n\big). \tag{4.26}$$

Similarly, taking the Wills polynomial $g_K(z)$ with roots ν_i, $i=1,\dots,n$, from the relation

$$g_K(z)=\sum_{i=0}^{n}\mathrm{V}_i(K)z^i=\mathrm{vol}(K)\prod_{i=1}^{n}(z-\nu_i)$$

we get

$$(-1)^i\frac{\mathrm{V}_{n-i}(K)}{\mathrm{vol}(K)}=\mathrm{s}_i\bigl(\nu_1,\dots,\nu_n\bigr). \tag{4.27}$$

Then from (4.26) and (4.27) we easily obtain the following relations between the roots of the Wills and the Steiner polynomials:

$$\begin{aligned}\mathrm{s}_i\bigl(\gamma_1^{-1},\dots,\gamma_n^{-1}\bigr)&=\kappa_i\,\mathrm{s}_i\bigl(\nu_1,\dots,\nu_n\bigr)\qquad\text{and}\\ \mathrm{s}_i\bigl(\nu_1^{-1},\dots,\nu_n^{-1}\bigr)&=\frac{\kappa_n}{\kappa_{n-i}}\,\mathrm{s}_i\bigl(\gamma_1,\dots,\gamma_n\bigr).\end{aligned}$$

However, just checking some easy examples, it can be seen that it is not possible to get relations of the type $\gamma_i=c(n)\nu_i$, for an n-dependent constant $c(n)$. Theorem 4.3.3 states a kind of asymptotic relation between them, which is a particular case of Theorem 4.2.2 for $p=2$.

Theorem 4.3.3 ([30]). *For $s\in\mathbb{N}$ fixed, let $K\in\mathcal{K}^s$ and let $\nu_1,\dots,\nu_s$ be the roots of $g_K(z)$. Embedding $K\subsetneq\mathbb{R}^n$, $n\geq s$, let $\gamma_{1,n},\dots,\gamma_{s,n}$ be the non-zero roots of $f_{K;B_n}(z)$. Then, reordering if necessary,*

$$\lim_{n\to\infty}\frac{\kappa_n}{\kappa_{n-1}}\gamma_{i,n}=\frac{\nu_i}{|\nu_i|^2},\qquad i=1,\dots,s.$$

Proof. We recall (Remark 4.5) that ν_i is a root of $g_K(z)$ if and only if $\nu_i/\left|\nu_i\right|^2=1/\overline{\nu_i}$ is a root of $f^g_{K;B_s}(z)=\sum_{i=0}^{s}\mathrm{V}_{s-i}(K)z^i$, $i=1,\dots,s$. Thus it suffices to show that (reordering if necessary)

$$\lim_{n\to\infty}\frac{\kappa_n}{\kappa_{n-1}}\gamma_{i,n}=\tilde{\nu}_i,\qquad i=1,\dots,s,$$

where $\tilde{\nu}_i$, $i=1,\dots,s$, are the roots of $f^g_{K;B_s}(z)$, which is a direct consequence of Theorem 4.2.2. □

In a sense, the above theorem says that for high dimension n, the Steiner polynomial $f_{K;B_n}(z)$ of a convex body K with fixed dimension $\dim K=s$ 'behaves as' its Wills polynomial $g_K(z)$.

Moreover, from Theorem 4.3.3 we immediately get the following corollary, which shows the asymptotic behavior of the (modulus and the argument of the) roots of the Steiner polynomial with respect to the ones of $g_K(z)$.

Corollary 4.3.1 ([30]). *Let $K\in\mathcal{K}^s$ and let $\nu_1,\dots,\nu_s$ be the roots of $g_K(z)$. Embedding $K\subsetneq\mathbb{R}^n$, $n\geq s$, let $\gamma_{1,n},\dots,\gamma_{s,n}$ be the non-zero roots of $f_{K;B_n}(z)$. Then the following properties hold:*

i) $\lim_{n\to\infty}|\gamma_{i,n}|=\infty$, $i=1,\dots,s$.

ii) Reordering if necessary, $\lim_{n\to\infty}\arg\gamma_{i,n}=\arg\nu_i$, $i=1,\dots,s$.

Proof. Property (ii) is straightforward.. For (i), using (4.22),

$$\lim_{n\to\infty}|\gamma_{i,n}| = \lim_{n\to\infty}\frac{\kappa_{n-1}}{\kappa_n}\frac{1}{|\nu_i|} \geq \lim_{n\to\infty}\sqrt{\frac{n}{2\pi}}\frac{1}{|\nu_i|} = \infty. \qquad \square$$

4.3.3 The roots of the Wills polynomial and other functionals

Here we consider the problem of relating the roots of the Wills polynomial of a convex body K with other functionals, namely, the in- and circumradius of K and the so-called successive minima of K with respect to the integer lattice $\mathbb{Z}^n$. We will start this subsection bounding the roots of the Wills functional in terms of the in- and circumradius.

Proposition 4.3.4 ([30]). *Let $K \in \mathcal{K}^n$. Then the roots ν_i, $i = 1, \dots, n$, of the Wills polynomial $g_K(z)$ are bounded by*

$$\frac{1}{\mathrm{V}_1(K)} \leq |\nu_i| \leq \frac{\mathrm{V}_{n-1}(K)}{\mathrm{vol}(K)}. \tag{4.28}$$

Both inequalities are sharp. In particular, we have

$$\frac{1}{2n}\frac{1}{\mathrm{R}(K)} \leq |\nu_i| \leq \frac{n}{2}\frac{1}{\mathrm{r}(K)}.$$

Proof. In order to bound the roots of $g_K(z)$, using Theorem 1.5.4, we have to find the minimum and maximum of $\mathrm{V}_j(K)/\mathrm{V}_{j+1}(K)$, $j = 0, \dots, n-1$. Writing this quotient via (1.9) in terms of the quermassintegrals, we get

$$\frac{\mathrm{V}_j(K)}{\mathrm{V}_{j+1}(K)} = \frac{j+1}{n-j}\frac{\kappa_{n-j-1}}{\kappa_{n-j}}\frac{\mathrm{W}_{n-j}(K)}{\mathrm{W}_{n-j-1}(K)}.$$

Aleksandrov-Fenchel inequalities (1.20) ensure that $\mathrm{W}_{n-j}(K)/\mathrm{W}_{n-j-1}(K)$ is increasing in j, and clearly $j+1$ is so. So we have to study the monotonicity of $\kappa_{n-j-1}/\big((n-j)\kappa_{n-j}\big)$ in j.

In order to do it, we consider the sequence $y_m = \kappa_{m-1}/(m\kappa_m)$. Then, by (1.16),

$$\frac{1}{m+1}\frac{\kappa_m}{\kappa_{m-2}} = \frac{1}{m+1}\frac{2\pi}{m} = \frac{1}{m}\frac{\kappa_{m+1}}{\kappa_{m-1}},$$

and using Aleksandrov-Fenchel inequalities (1.20) for $\kappa_m = \mathrm{W}_m(C_n)$, we get

$$y_{m+1} = \frac{1}{m+1}\frac{\kappa_m}{\kappa_{m+1}} = \frac{1}{m}\frac{\kappa_{m-2}}{\kappa_{m-1}} \leq \frac{1}{m}\frac{\kappa_{m-1}}{\kappa_m} = y_m.$$

Therefore, y_m is a decreasing sequence in m, i.e., $\kappa_{n-j-1}/\big((n-j)\kappa_{n-j}\big)$ is an increasing sequence in j. Thus, altogether we get

$$\frac{1}{\mathrm{V}_1(K)} = \frac{1}{n}\frac{\kappa_{n-1}}{\mathrm{W}_{n-1}(K)} \leq \frac{\mathrm{V}_j(K)}{\mathrm{V}_{j+1}(K)} \leq \frac{n}{2}\frac{\mathrm{W}_1(K)}{\mathrm{W}_0(K)} = \frac{\mathrm{V}_{n-1}(K)}{\mathrm{vol}(K)}$$

for $j = 0, \dots, n-1$, which shows (4.28).

We notice that for $n = 1$, any line segment gives equality in both inequalities. For arbitrary dimension, let $Q(\ell)$ be the n-dimensional orthogonal box with edge-lengths $1, \ell, \dots, \ell$, $\ell \geq 1$, for which $\mathrm{V}_i\big(Q(\ell)\big) = \mathrm{s}_i\big(1, \ell, \dots, \ell\big)$ and $\nu_1 = 1$ is one of the roots of $g_{Q(\ell)}(z)$ (see Remark 4.6). Then

$$\lim_{\ell\to\infty} \frac{\mathrm{V}_{n-1}\big(Q(\ell)\big)}{\mathrm{vol}\big(Q(\ell)\big)} = \lim_{\ell\to\infty} \frac{\ell^{n-1} + (n-1)\ell^{n-2}}{\ell^{n-1}} = 1 = |\nu_1|,$$

which shows that the upper bound is sharp. Analogously, taking $\bar{Q}(\ell)$ the n-dimensional orthogonal box with edge-lengths $1, \ell, \dots, \ell$, $\ell \leq 1$, then

$$\lim_{\ell\to 0} \frac{1}{\mathrm{V}_1\big(\bar{Q}(\ell)\big)} = \lim_{\ell\to 0} \frac{1}{(n-1)\ell + 1} = 1 = |\nu_1|,$$

which shows that the lower bound is sharp.

The bounds in terms of the in- and circumradius follow immediately from (1.12) (via (1.9)), taking into account that $\kappa_{n-1}/\kappa_n \geq 1/2$ for all $n \geq 1$. □

For the next proposition, we need to deal with a special kind of sets: the tangential bodies of a ball (see Definition 1.11 and Remark 1.2).

Proposition 4.3.5 ([30]). *Let $K \in \mathcal{K}^n$ and let ν_i, $i = 1, \dots, n$, be the roots of the Wills polynomial $g_K(z)$. If $\mathrm{Re}\,\nu_i = -a$, $a > 0$, for all $i = 1, \dots, n$, then*

$$\frac{1}{2\mathrm{R}(K)} \leq a \leq \frac{1}{2\mathrm{r}(K)}.$$

Equality holds in the right inequality if and only if K is a tangential body of the ball $\mathrm{r}(K)B_n$.

Proof. Using (4.27) for $i = 1$ and (1.9), we have

$$-na = \sum_{i=1}^{n} \mathrm{Re}\,\nu_i = \sum_{i=1}^{n} \nu_i = -\frac{\mathrm{V}_{n-1}(K)}{\mathrm{vol}(K)} = -\frac{n}{2}\frac{\mathrm{W}_1(K)}{\mathrm{W}_0(K)},$$

and thus, by (1.12),

$$\frac{1}{2\mathrm{R}(K)} \leq a \leq \frac{1}{2\mathrm{r}(K)}.$$

Finally, equality $a = 1/\big(2\mathrm{r}(K)\big)$ holds if and only if we have equality in $\mathrm{W}_0(K) \geq \mathrm{r}(K)\mathrm{W}_1(K)$, i.e., when K is a tangential body of $\mathrm{r}(K)B_n$ (see Theorem 1.3.1 and Remark 1.2). □

Proposition 4.3.5 contrasts with the case of the Steiner polynomial, where only the one of the ball can have all its roots with equal real part (in fact, it has an n-fold real root).

Remark 4.8. *From the above argument we also notice that*

$$\frac{n}{2\mathrm{R}(K)} \leq \big|\mathrm{Re}\,\nu_1 + \cdots + \mathrm{Re}\,\nu_n\big| \leq |\mathrm{Re}\,\nu_1| + \cdots + |\mathrm{Re}\,\nu_n|.$$

In [61] Wills studied relations between the roots of the Wills polynomial of a 0-symmetric convex body and its successive minima. Here we slightly improve some of those relations.

Proposition 4.3.6 ([30]). *Let $K \in \mathcal{K}^n$ be 0-symmetric and let ν_i, $i = 1, \dots, n$, be the roots of the Wills polynomial, ordered such that $|\nu_1| \leq \cdots \leq |\nu_n|$. Then:*

i) $\lambda_{i+1}(K) \dots \lambda_n(K) < 2^{n-i}\binom{n}{i}|\nu_{i+1}| \dots |\nu_n|$, $i = 1, \dots, n-1$.

ii) $\lambda_n(K) + (n-1)\mathrm{r}(K)^{n-1}/\,\mathrm{R}(K)^n \leq -2(\nu_1 + \cdots + \nu_n)$.

Equality holds in (ii) if and only if $K = B_n$.

It improves items (d) and (b) of Theorem 1 in [61], respectively.

Proof. In [24] the following sharp inequality was proved for a 0-symmetric convex body $K \in \mathcal{K}^n$:

$$\lambda_{i+1}(K) \dots \lambda_n(K)\mathrm{vol}(K) < 2^{n-i}\mathrm{V}_i(K),$$

$i = 1, \dots, n-1$. This, together with (4.27), gives, on the one hand,

$$\lambda_{i+1}(K) \dots \lambda_n(K) < 2^{n-i}(-1)^{n-i}\mathrm{s}_{n-i}\big(\nu_1, \dots, \nu_n\big) \leq 2^{n-i}\binom{n}{i}|\nu_{i+1} \dots \nu_n|.$$

On the other hand, the known Wills conjecture, proved independently by Bokowski and Diskant, states that (see e.g. page 389 of [52] and the references inside)

$$\mathrm{vol}(K) - \mathrm{r}(K)\mathrm{S}(K) + (n-1)\kappa_n\mathrm{r}(K)^n \leq 0.$$

Taking into account that $\lambda_n(K) \leq 1/\mathrm{r}(K)$, because $K \supset \mathrm{r}(K)B_n$, and also that $\mathrm{vol}(K) \leq \kappa_n\mathrm{R}(K)^n$ (cf. (1.12)), using (4.27) we get the required inequality:

$$\begin{aligned} -2\sum_{i=1}^{n}\nu_i &= 2\frac{\mathrm{V}_{n-1}(K)}{\mathrm{vol}(K)} = \frac{\mathrm{S}(K)}{\mathrm{vol}(K)} \geq \frac{1}{\mathrm{r}(K)} + (n-1)\frac{\kappa_n}{\mathrm{vol}(K)}\mathrm{r}(K)^{n-1} \\ &\geq \lambda_n(K) + (n-1)\frac{\mathrm{r}(K)^{n-1}}{\mathrm{R}(K)^n}. \end{aligned}$$

Since equality in Wills' conjecture holds if and only if K is the Euclidean ball, we obtain the same characterization for the equality case in (ii). □

4.3.4 A brief note on the Wills polynomial of the ball

The Wills polynomial of the ball satisfies the nice property (see identity (4.4) in [60])

$$i!\,\kappa_i\,g_{B_n}^{(n-i)}(z) = n!\,\kappa_n\,g_{B_i}(z). \tag{4.29}$$

We have also proved that the Wills polynomial of the ball determines the cone of roots, i.e., $\mathcal{R}^g(n; B_n) = \mathcal{R}^g(B_n; B_n)$, for dimensions $n = 2, 3$. In this section we show some additional properties of this particular polynomial $g_{B_n}(z)$ and the cone $\mathcal{R}^g(B_n; B_n)$.

Proposition 4.3.7 ([30]). *The Wills polynomial $g_{B_n}(z)$ is weakly stable for $n \leq 13$ and it is not for $n \geq 14$. Moreover, $\mathcal{R}^g(B_{n-1}; B_{n-1}) \subsetneq \mathcal{R}^g(B_n; B_n)$ if $n \leq 14$.*

Proof. Applying the stability criterion given by Theorem 1.5.2, it is easy to check that $g_{B_n}(z)$ is weakly stable for $n \leq 13$, whereas the polynomial $g_{B_{14}}(z)$ has a root with positive real part ($\nu \approx 0.04562 + 1.81036\mathrm{i}$).

Let $n \geq 14$ be any positive integer such that the polynomial $g_{B_n}(z)$ is not weakly stable. If we assume that $g_{B_{n+1}}(z)$ is weakly stable, then we have $\operatorname{conv}\bigl\{\nu : g_{B_{n+1}}(\nu) = 0\bigr\} \subsetneq \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$. The well-known Gauss-Lucas theorem states that all roots of the derivative of a non-constant polynomial lie in the convex hull of the set of roots of the polynomial (see Theorem 1.5.5). This result together with the fact $g'_{B_n}(z) = (n\kappa_n/\kappa_{n-1})g_{B_{n-1}}(z)$ (cf. (4.29)) shows that $g_{B_n}(z)$ is weakly stable, a contradiction. So, $g_{B_{n+1}}(z)$ is also weakly stable.

On the other hand, let $\bar{A}$ denote the set of conjugates of complex numbers in $A \subset \mathbb{C}^+$. Because of the (weak) stability of $g_{B_n}(z)$, the cone $\mathcal{R}^g(B_n; B_n) \cup \overline{\mathcal{R}^g(B_n; B_n)}$ is convex for $n < 14$, and then it contains the set $\operatorname{conv}\bigl\{\nu : g_{B_n}(\nu) = 0\bigr\}$. Again, Gauss-Lucas' theorem together with (4.29) prove that $\mathcal{R}^g(B_{n-1}; B_{n-1}) \subset \mathcal{R}^g(B_n; B_n)$, $n < 14$. Numerical computations give the strict inclusion. Finally, the non-stability of $g_{B_{14}}(z)$ concludes the proof. □

We finish the chapter by showing the following result.

Proposition 4.3.8. *Let $\nu_1, \dots, \nu_n$ be the roots of the Wills polynomial $g_{B_n}(z)$, and let $K \in \mathcal{K}^n$ be such that its Steiner polynomial $f_{K;B_n}(z)$ has only (zero and non-zero) real roots $\gamma_1, \dots, \gamma_n$. Then for all $\nu \in \mathbb{C}^+$ with $g_K(\nu) = 0$ the following properties hold:*

i) If $n \geq 14$, $\nu \in \lambda \operatorname{conv}\{\nu_1, \dots, \nu_n\}$, with $\lambda = \max\bigl\{|\gamma_1|, \dots, |\gamma_n|\bigr\}$.

ii) If $n < 14$, $\nu \in \mathcal{R}^g(B_n; B_n)$. Moreover, $\mathcal{R}^g(B_n; B_n)$, $n < 14$, contains the roots of the Wills polynomials $g_{\bar{K}}(z)$ of all convex bodies $\bar{K} \in \mathcal{K}^s$, $s \leq n$, such that $f_{\bar{K};B_s}(z)$ has only real roots.

Proof. We notice that if $f_{K;B_n}(z)$ has only real roots, then $\tilde{f}(z) = \sum_{j=0}^{n} \binom{n}{j} \mathrm{W}_j(K) z^{n-j}$ also does it.

First we show i). Since $g_{B_n}(z)$ is not weakly stable for $n \geq 14$, there exists one root of $g_{B_n}(z)$, say ν_1, such that $\operatorname{Re} \nu_1 \geq 0$. Moreover, since

$$-\sum_{i=1}^{n} \operatorname{Re} \nu_i = -\sum_{i=1}^{n} \nu_i = \frac{n}{2} > 0$$

(cf. (4.27)), there exists another root of $g_{B_n}(z)$, say ν_2, with $\operatorname{Re} \nu_2 < 0$. Then $0 \in \operatorname{conv}\{\nu_1, \dots \nu_n\}$ and hence $\operatorname{conv}\{\nu_1, \dots \nu_n\}$ is the intersection of half-planes $H_{c_j} = \bigl\{x \in \mathbb{R}^2 : \langle x, u_j \rangle \leq c_j\bigr\}$, with $c_j > 0$. Theorem 1.5.6 applied to each H_{c_j} and to the polynomials $(1/\kappa_n)g_{B_n}(z)$, $\tilde{f}(z)$ and $g_K(z)$, ensure that all roots of $g_K(z)$ lie in $H_{\lambda c_j}$, for all j, and thus also in $\lambda \operatorname{conv}\{\nu_1, \dots \nu_n\}$.

For ii) we just have to apply again Theorem 1.5.6 to the polynomials $(1/\kappa_n)g_{B_n}(z)$, $\tilde{f}(z)$ and $g_K(z)$, and to the two half-planes determining the convex cone $\mathcal{R}^g(B_n;B_n) \cup \overline{\mathcal{R}^g(B_n;B_n)}$, $n<14$, to obtain that every root of $g_K(z)$ is of the form $\nu = rw$ for $r \in \mathbb{R}$ and $w \in \mathcal{R}^g(B_n;B_n) \cup \overline{\mathcal{R}^g(B_n;B_n)}$. The last assertion arises from the facts that any Wills polynomial is invariant with respect to the embedding dimension and that $\mathcal{R}^g(B_s;B_s) \subsetneq \mathcal{R}^g(B_n;B_n)$ (Proposition 4.3.7). □

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