Universal Distribution Functions in Two-Dimensional Localized Systems

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(Received 1 March 2007; published 11 September 2007)

We find the conductance distribution function of the two-dimensional Anderson model in the strongly localized limit. The fluctuations of $\ln g$ grow with lateral size as $L^{1/3}$ and follow a universal distribution that depends on the type of leads. For narrow leads, it is the Tracy-Widom distribution, which appears in the problem of the largest eigenvalue of random matrices from the Gaussian unitary ensemble and in many other problems like the longest increasing subsequence of a permutation, directed polymers, or polynuclear growth. We also show that for wide leads the conductance follows a related, but different, distribution.

DOI: 10.1103/PhysRevLett.99.116602 PACS numbers: 72.20.−i, 71.23.An

The distribution function of the conductance g of disordered systems is very well understood in the metallic regime, but poorly understood in the localized phase. Experimental measurements of coherent transport at low temperatures are difficult in the strongly localized regime. However, knowledge of the zero temperature conductance distribution is of interest to better understand variable range hopping conductance, the metal-insulator transition in three dimensions or the crossover between the diffusive and the localized regime in two dimensions.

In one-dimensional systems, it has been shown that all the cumulants of $\ln g$ scale linearly with system size [1]. Thus, the distribution function of $\ln g$ approaches a Gaussian form for asymptotically long systems and is fully characterized by two parameters, the mean $\langle \ln g\rangle$ and the variance $\sigma^2 = \langle \ln^2 g\rangle - \langle \ln g\rangle^2$. Both parameters are related to each other, supporting the extension of the single parameter scaling (SPS) hypothesis [2] to the distribution function of the conductance [3].

In higher dimensions, it is far more difficult to do analytical calculations and numerical simulations have been limited until recently to small sample sizes. In the strong localization regime, the general belief was that $\ln g$ should be normally distributed and the variance would depend linearly on size both in two-dimensional (2D) and three-dimensional (3D) systems [4–6]. In contrast, we found numerically that the variance behaves as [7,8]

$$\sigma^2 = A\langle -\ln g\rangle^\alpha + B, \tag{1}$$

with the exponent α equal to $2/3$ in 2D and $2/5$ in 3D systems. Also, we showed that the skewness of the distribution of $\ln g$ does not tend to zero in the highly localized regime [9], demonstrating that the distribution is not Gaussian (see also Ref. [10]). The constants A and B in Eq. (1) are model or geometry dependent. The precise knowledge of the dependence of σ^2 with $\langle -\ln g\rangle$ made much easier the numerical verification of the SPS hypothesis [7].

Nguyen *et al.* (NSS) [11] proposed a simplification to the Anderson model, convenient for numerical purposes. Their model accounts for quantum interference effects in the localized regime, where the tunneling amplitude between two sites is calculated considering only the shortest or forward-scattering paths. Medina and Kardar [12] studied in detail the NSS model and computed numerically the distribution of the logarithm of the tunneling probability and found that its variance increases with distance as $r^{2/3}$ for 2D systems. They also established an analogy of the NSS model with directed polymers, where this result is well-known. The applicability of their results to the conductance of general disordered systems with finite localization lengths was not clear. In the last decade, there have been important advances in our understanding of the distribution function of directed polymers and related problems. For a specific directed polymer model, the distribution function of the lowest energy state was obtained exactly [13] in terms of the Tracy-Widom (TW) distribution, which was originally obtained as the distribution of the largest eigenvalue of random matrices belonging to the Gaussian unitary ensemble [14]. The TW distribution also appears in the fluctuations of the length of the longest common subsequence in a random permutation [15] and in the polynuclear growth model [16], which is closely related to the problem of the height function in the Kardar-Parisi-Zhang equation [17].

In this Letter we study the distribution function of $\ln g$ for disordered 2D systems. First, we prove through a mapping to the model solved by Johansson [13] that for a specific type of disorder it is given by the TW distribution. Second, we show numerically that the same distribution also applies to the Anderson model in the strongly localized regime. Third, we demonstrate that boundary conditions change the distribution function.

We focus on the Anderson model on a square sample of finite size $L \times L$ described by the Hamiltonian

$$H = \sum_i \epsilon_i a_i^\dagger a_i + t\sum_{i,j} a_j^\dagger a_i + \text{H.c.}, \tag{2}$$

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where the operator $a_i^\dagger$ (a_i) creates (destroys) an electron at site i of an square lattice and ϵ_i is the energy of this site chosen at random from a given distribution. The double sum runs over nearest neighbors. The hopping matrix element t is taken equal to 1, which set the energy scale, and the lattice constant equal to 1, setting the length scale. The unit of conductance is $2e^2/h$.

One can write the matrix elements of the Green function between two sites a and b in terms of the locator expansion

$$\langle a|G|b\rangle = \sum_{\Gamma} \prod_{i\in\Gamma} \frac{1}{E - \epsilon_i}, \tag{3}$$

where the sum runs over all possible paths connecting the two sites a and b. In general the convergence of this series is very problematic, but in the strongly localized regime for distances much larger than the localization length one expects that the previous sum is dominated by the forward-scattering paths. Back-scattering paths should be irrelevant in the renormalization-group sense. Based on this idea, the NSS model only considers directed paths between two points in opposite corners of a square lattice and a site disorder energy with only two possible values, W and $-W$, trying to maximize interference effects.

The calculation of the quantum amplitude between two points in the approximation of forward-scattering paths is formally similar to the calculation of the partition function of directed polymers in a random potential

$$Z = \sum_{\Gamma} \exp\left\{-\beta \sum_{i\in\Gamma} h_i\right\}, \tag{4}$$

where $\beta = 1/kT$, h_i is a random site energy, and Γ runs over all possible configurations of the directed polymer. Equations (3) and (4) are equivalent provided that we can identify $-\beta h_i$ with $\ln(E - \epsilon_i)$. If we require the disorder energies h_i to be real then all the values of $E - \epsilon_i$ have to be positive. In this case, we can map the distribution of $\ln g$ in our system to the distribution of the free energy in directed polymers.

Johansson [13] was able to obtain exactly the asymptotic distribution function of the ground state energy H for a specific type of disordered polymer. In his model the random site energies take integer values with probabilities $\Pr(h_i = k) = (1-p)p^k$. He showed that the ground state energy for polymers running between the origin and the point (x, y) is given by

$$H(x, y) \to \frac{2\sqrt{pxy} + p(x+y)}{1-p} + \frac{(pxy)^{1/6}}{1-p}\left[(1+p) + \sqrt{\frac{p}{xy}}(x+y)\right]^{2/3} \chi_2 \tag{5}$$

where χ_2 is a random variable with the TW distribution, corresponding to the distribution of the largest eigenvalue of a complex Hermitian random matrix [14]. χ_2 verifies $\Pr(\chi_2 < x) = F_2(x) = e^{-g(x)}$ where $g''(x) = u^2(x)$, $g(x) \to 0$ as $x \to \infty$, and $u(x)$ is the global positive solution of the Painlevé II equation $u'' = 2u^3 + xu$ with $u(x) \to \mathrm{Ai}(x)$ as $x \to \infty$, $\mathrm{Ai}(x)$ being the Airy function.

The mapping of Johansson's model to the localization problem corresponds to a distribution of disorder energies with values $\epsilon_i - \epsilon = e^{-\beta h_i}$ with probability $\Pr(h_i = k) = (1-p)p^k$ for $k = 0, 1, 2, \ldots$, in the limit $\beta \to \infty$. This is a very specific model, but we expect that their results apply in a much more general context, as already suggested by Johansson [13]. In this model, the conductivity is likely to be dominated by the most favorable forward-scattering path, and one can question the applicability of the results to a uniform disorder or even more to the NSS model, which cannot be mapped to the directed polymers due to alternating signs of the disorder energies. We note that in the NSS model all trajectories have the same amplitude and interference effects between different paths is maximized. However, our previous results for both the NSS and the Anderson models showed that the cumulants of $\ln g$ are given by

$$\kappa_j = A_j L^{j/3} + B_j \tag{6}$$

for $j > 1$ [8]. The mean, $\langle \ln g\rangle$, also verifies this equation, but with an extra term $2L/\xi$, where ξ is the localization length. The exponents in this equation are the same as those implied by the distribution in Eq. (5). It is then interesting to check if the distribution function given by Eq. (5) also applies to the Anderson and the NSS models.

We have obtained numerically the distribution function of $\ln g$ for the Anderson and the NSS models in 2D samples. For the Anderson model we have calculated the conductance through Landauer's formula in terms of the transmission between perfect leads. This is obtained from the Green function, which can be calculated propagating layer by layer with the recursive Green function method [18]. We have considered ranges of disorder W equal to 13, 15, and 25, which correspond to localization lengths of 1.12, 2.4, and 3.2, respectively, and lateral dimensions up to $L = 200$. The number of different realizations for a given disorder and size is larger than 6×10^5 in all cases. We have considered two types of leads connected to our $L \times L$ disordered region: (i) narrow one-dimensional leads attached at the center of opposite edges and (ii) wide leads with the same section as the samples. In both cases, they are represented by the same Hamiltonian as the system, Eq. (2), but without diagonal disorder. We use cyclic periodic boundary conditions in the direction perpendicular to the leads.

In Fig. 1 we plot histograms of $\ln g$ for the Anderson model as a function of $\chi = (\ln g - A)/B$, where A and B are chosen in order to have the same mean and variance as the theoretical distributions. The data are for narrow (solid symbols) and wide leads (empty symbols) and several sizes and ranges of the disorder: $W = 25$ and $L = 100$ (circles), $W = 15$ and $L = 150$ (squares), and $W = 13$ and $L = 200$

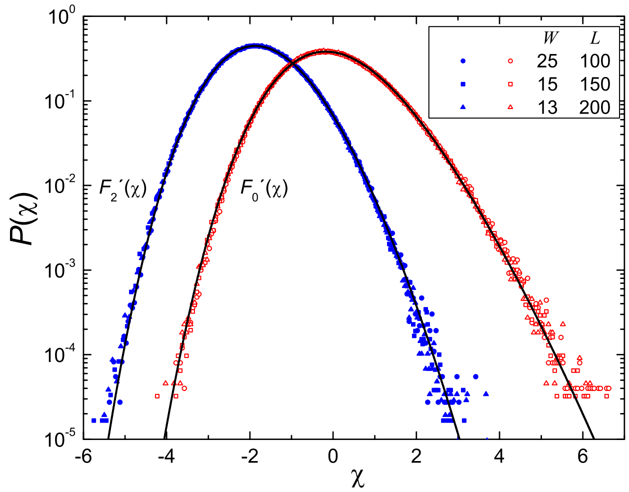

FIG. 1 (color online). Histograms of $\ln g$ versus the scaled variable χ for several sizes and disorders of the Anderson model with narrow (solid symbols) and wide (empty symbols) leads. The continuous lines correspond to $F_2'(\chi)$ and $F_0'(\chi)$.

(triangles). Let us concentrate first in the case of narrow leads, since it is similar to the directed polymer model, represented by solid symbols in Fig. 1. The solid line on the left of this figure corresponds to the TW distribution, whose accumulated distribution is F_2. We see a perfect agreement between our numerical results and the TW distribution for more than 4 orders of magnitude. A similar agreement is found for the NSS model (not shown). Considering the size dependence of the mean and the variance of $\ln g$ and the excellent agreement between our data and the TW distribution, we conclude that in the strongly localized regime

$$\ln g = -\frac{2L}{\xi} + \alpha\left(\frac{L}{\xi}\right)^{1/3}\chi_2, \qquad (7)$$

where α is a constant and χ_2 a random variable with the TW distribution.

In the SPS regime α is a constant, independent of the disorder, the system size or the Fermi energy. We found, from present results and previous calculations on the behavior of the variance, that it is approximately equal to 3.4 for the Anderson model with narrow leads. Equation (7) must be also valid outside the SPS regime (when the localization length is very small or when the Fermi energy lies in the band tails), but with a nonuniversal value of α. The data set in Fig. 1 for $W = 25$ correspond to a localization length $\xi = 0.97$, of the order of the lattice spacing, and fit the TW distribution very well. The value of α is 3.5 in this case.

The TW distribution has a mean $\langle\chi\rangle = -1.77109$. According to Eq. (7), this implies a contribution to $\langle\ln g\rangle$ proportional to $L^{1/3}$, already observed by us [9] in the Anderson model with narrow leads. However, we did not find any such contribution for wide leads, which constitutes a strong indication that the conductance distribution may depend on the leads, even in the strongly localized regime. We can expect a situation similar to polynuclear growth models [16], where the height distribution was found to depend on the initial conditions. Our problem with narrow leads is directly related to the droplet model in Ref. [16], which starts from an initial preferential point. With wide leads we have translational invariance and all the initial (and final) points are equivalent, a problem similar to stationary growth. In this case, the height fluctuations are described by the accumulated distribution [16]

$$F_0(x) = [1 - (x + 2f'' + 2g'')g']e^{-(g+2f)}, \qquad (8)$$

where $f'(x) = -u(x)$, with $f(x) \to 0$ for $x \to \infty$, and $g(x)$ and $u(x)$ defined as above. This new distribution has mean equal to zero, as required in our case in order not to have $L^{1/3}$ contributions to $\langle\ln g\rangle$. In Fig. 1 we show the histograms of $\ln g$ for the Anderson model with wide leads (empty symbols) for several disorders and sizes. The full line on the right of the figure corresponds to the derivative of $F_0(x)$. The agreement between the numerical data and the theoretical distribution is again excellent, showing that $\ln g$ satisfies in this case

$$\ln g = -\frac{2L}{\xi} + \beta\left(\frac{L}{\xi}\right)^{1/3}\chi_0, \qquad (9)$$

where β is a constant and χ_0 a new random variable given by $\Pr(\chi < x) = F_0(x)$. In the SPS regime $\beta \approx 2.2$. Equations (7) and (9) constitute our main result.

Our results fully suggest that the Anderson and related directed path models, in 2D in the strongly localized regime, will verify an equation of the form (7) or (9) for any range of parameters, type of disorder, geometry or boundary conditions. The distribution of the random variable depends on the boundary conditions. Nevertheless, there are some results which are very robust, like the exponent of 1/3 in Eq. (7) and (9) characterizing the size of fluctuations. Its value determines the behavior of the cumulants, Eq. (6), and, as we will see, of the tails of the distribution. From Eq. (6), we expect $\langle g^n\rangle$ to be a function of the variable $n^{1/3}L$, beside the contribution of the mean proportional to L, in the large L limit

$$\ln\langle g^n\rangle \approx -\frac{2Ln}{\xi} + \sum_{j=1}^{\infty}\frac{n^j A_j L^{j/3}}{j!}. \qquad (10)$$

Since the maximum of $\langle g^n\rangle$ is reached in the absence of interference, and the number of directed paths grows exponentially with L, the sum in Eq. (10) cannot grow faster than L. As this sum is a function of $nL^{1/3}$ we expect a dominant contribution proportional to n^3L for large L, and so

$$\ln\langle g^n\rangle \approx -\frac{2Ln}{\xi} + \rho n^3 L. \qquad (11)$$

This approximation was already proposed by Medina and Kardar [12] in the context of the NSS model. The distribution of $\ln g$ is the inverse Laplace transform of $\langle g^n \rangle$ and from Eq. (10) and the saddle point approximation we obtain for the high conductance tail of the distribution a behavior of the type $\ln P(\ln g) \propto z^{3/2}$, where $z = (\ln g - 2L/\xi)/L^{1/3}$. Indeed, for the two types of leads studied $-\ln F_j'(x) = d_j x^{3/2}/3$ for $x \to \infty$ with $d_0 = 2$ and $d_2 = 4$. Equation (11) is valid for the high conductance tail only. The other tail might be more sensitive to boundary conditions, although both distributions F_0 and F_2 behave as $-\ln F_j(x) = |x|^3/12$ for $x \to -\infty$ [16].

Present results confirm our previous belief that, in the strongly localized regime, directed path models are in the same universality class as the Anderson model [8,11,12]. While the NSS model pretended to maximize interference effects, Johansson's model only considers the most important path. The agreement between both models indicates that it is percolation and not interference the dominant effect in this regime. We expect that the main effect of interference between different paths is a renormalization of the disorder energies. This information may be relevant to deal with interacting systems.

The knowledge of the distribution functions, Eqs. (7) and (9), is of practical interest for the calculation of the localization length. Equation (7) incorporates contributions to the mean proportional to $L^{1/3}$, which can be relatively important for the typical sizes studied. Neglecting this term can cause errors of the order of 20% in the estimate of the localization length.

We have checked that for $L/\xi \gtrsim 6$ the conductance distribution is fitted by Eqs. (7) and (9) much better than by a log-normal. For mesoscopic samples close to the condition $L/\xi \approx 6$ it should be possible to test experimentally our predictions, since $g \approx g_0 \exp(-12)$ where g_0 is close to 1 in units of $2e^2/h$. Our results can also be verified through the behavior of the cumulants of the distribution. Equations (7) and (9) predict universal values for the skewness, kurtosis, etc., of the distribution. These limiting values can be obtained from a measurement of the cumulants in any range of parameter in the localized region, since Eq. (6) it is fairly well verified even near the crossover. Thus, from the tendency of the second and third cumulants it is possible to derive the asymptotic value for the skewness: $A_3/A_2^{3/2} = 0.359$ (for wide leads).

In 3D systems, the variance is proportional to $\langle \ln g \rangle^{2/5}$ [8], so we expect the conductance to be distributed according to $\ln g = 2L/\xi + \alpha_3 (L/\xi)^{1/5} \chi$, being χ an unknown random variable. The arguments given above for the high conductance tail predict a behavior of the form $\ln P(\ln g) \propto (\ln g)^{5/4}$, in agreement with preliminary numerical results.

In summary, we have found the distribution functions of the conductance of the two-dimensional Anderson model in the strongly localized regime for two types of leads. The distributions obtained are the TW distribution and a related one, and they appear in many other problems. Our results, Eqs. (7) and (9), are fully consistent with the strong version of the SPS [19] if we take into account that boundary conditions (leads) are not irrelevant variables, in the renormalization-group sense, and may change the universality class.

The authors would like to acknowledge financial support from the Spanish DGI, Project No. FIS2006-11126, and Fundación Seneca, Project No. 03105/PI/05.

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