A LIMITING CASE OF ULTRASYMMETRIC SPACES

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Abstract. We study ultrasymmetric spaces in the case in which the fundamental function belongs to a limiting class of (quasi)-concave functions. In the process we study limiting cases of J interpolation spaces and establish new J-K identities as well as a reiteration theorem for these limiting interpolation methods.

1. Introduction

Rearrangement invariant (symmetric) function spaces is a class of spaces widely studied and very useful in applications. The best known are Lebesgue spaces L_p, Lorentz spaces $L_{p,q}$, Lorentz-Zygmund spaces $L_{p,q}(\log L)^{\alpha}$, Orlicz spaces and their generalizations. We refer, for example, to the books by Bennett and Sharpley [3], Kreĭn, Petunin and Semenov [17], Lindenstrauss and Tzafriri [18] for more information about symmetric spaces.

In 2003 E. Pustylnik introduced a large class of spaces which comprises an important part of symmetric spaces that includes Lorentz-Zygmund spaces and their generalizations. These spaces, called *ultrasymmetric spaces*, are symmetric spaces with the additional property that they are interpolation spaces for the couple $(\Lambda_\varphi, M_\varphi)$ formed by the Lorentz and the Marcinkiewicz spaces with the same fundamental function φ. That is to say, they are invariant for any quasilinear operator T which is bounded on Λ_φ and on M_φ. See [21] for more information.

An important advantage of ultrasymmetric spaces is that they have a simple *analytical description.* If G is an ultrasymmetric space the norm of the elements of G have the form

$$\|f\|_G \sim \|\varphi(t) f^*(t)\|_{\widetilde{E}},$$

where $\widetilde{E}$ is a symmetric function space with respect to the measure dt/t on $(0,\infty)$, φ is the fundamental function of the space G and f^* is the decreasing rearrangement of f. This allows generalizations of known results about classical spaces and turns out to be useful in applications. For example, in [22], Pustylnik generalized the results of Pietsch [20] on approximation spaces by modelling the sequence of approximation numbers in ultrasymmetric sequence spaces instead of on Lebesgue sequence spaces. See also [21] and [23] for other applications.

However, the analytical description holds only for those ultrasymmetric spaces that are not too "close" to L_∞. That is to say, the fundamental function of the

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space G does not grow too slowly (both *extension indices* have to be strictly greater that zero).

Our goal in this paper, in contrast to the work of Pustylnik, is to give an analytical description for precisely those ultrasymmetric spaces that are close to L_∞, in the sense that the extension indices of the fundamental function are both equal to zero.

An easy example that illustrates this situation is the case of Lorentz-Zygmund spaces $L_{\infty,q}(\log L)^\alpha$. These are the spaces of all measurable functions on $(0,1)$ defined through the norms

$$\|f\|_{\infty,q,\alpha} = \left(\int_0^1 \left((\ln \tfrac{e}{t})^\alpha f^*(t) \right)^q \frac{dt}{t} \right)^{1/q}, \quad \text{if } \alpha + \frac{1}{q} < 0$$
$$\|f\|_{\infty,\infty,\alpha} = \sup_{0<t<1} (\ln \tfrac{e}{t})^\alpha f^*(t), \qquad \text{if } \alpha < 0.$$

The fundamental function of the space $L_{\infty,q}(\log L)^\alpha$, $1 \leq q \leq \infty$, is a power of the logarithm with both extension indices equal to zero, so the results in [21] do not cover this case that requires a special discussion. In Section 4 we solve this problem and we obtain that $L_{\infty,q}(\log L)^\alpha$ are ultrasymmetric spaces when $\alpha + \frac{1}{q} < 0$.

The refinement of the known techniques to characterize ultrasymmetric spaces which are too close to L_∞ needs some new results in *interpolation theory*. In particular, we make use of limiting J-spaces and establish an equivalence theorem between limiting K and J constructions. The equivalence between limiting K and J-spaces has been widely studied in recent years. For ordered couples, see the papers by Cobos and Kühn [9], Cobos, Fernández-Cabrera, Kühn and Ullrich [6] and Cobos, Fernández-Cabrera and Mastylo [7]. For arbitrary Banach couples and some choices of parameters, see the papers by Cobos, Segurado [10, 11] and Cobos, Fernández-Cabrera and Silvestre [8]. Our approach overcomes these restrictions allowing general interpolation couples and a wider family of interpolation parameters as we present in Section 2. The limiting equivalence theorem let us in a position to establish a new reiteration result, Theorem 3.6, that will turn out to be the cornerstone for the analytical characterization of the limiting ultrasymmetric spaces.

The paper is organized as follows. In Section 2 we define the family of concave functions we shall work with and we describe limiting K and J-spaces. Section 3 is devoted to prove equivalence and reiteration theorems. Finally, in Section 4 we characterize Lorentz and Marcinkiewicz spaces whose fundamental functions lie in the limiting class of concave functions and we establish the analytical description of ultrasymmetric spaces.

Throughout the paper we do not distinguish spaces with equivalent norms. Given two functions f and g defined on $(0,\infty)$, by $f \lesssim g$ we mean that there is a contant $C > 0$, independent of all parameters, such that $f(t) \leq Cg(t)$ for all $t > 0$. We write $f \sim g$ if $f \lesssim g$ and $g \lesssim f$.

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2. Preliminaries

In this section we collect some of the definitions and results that are needed to follow the paper. Our framework is the family of function spaces on $(0,\infty)$, with the Lebesgue measure, having the Fatou property as described in [3]. Among these, rearrangement invariant (r.i.) spaces are characterized by the fact that equidistributed functions have the same norm. This property allows to define the fundamental function of an r.i. space, E, as $\varphi_E(t) = \|\chi_D\|_E$, where $\mathrm{mes}(D) = t$, which is a quasi-concave function. Since E can be equivalently renormed with a r.i. norm in such a way that the resulting fundamental function is concave, we will assume that φ_E is concave.

Furthermore, associated to the concave function φ_E we have the Lorentz space Λ_{φ_E} and the Marcinkiewicz space M_{φ_E} defined by the norms

$$\begin{aligned} \|f\|_{\Lambda_{\varphi_E}} &= \int_0^\infty f^*(t)\, d\varphi_E, \\ \|f\|_{M_{\varphi_E}} &= \sup_{0<t<\infty} \varphi_E(t) f^{**}(t), \end{aligned}$$

where $f^*(t) = \inf\{\lambda > 0,\ \mathrm{mes}\{|f| > \lambda\} \le t\}$ and $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$, $t > 0$.

These spaces are, respectively, the smallest and largest r.i. spaces with fundamental function φ_E, that is

$$\Lambda_{\varphi_E} \hookrightarrow E \hookrightarrow M_{\varphi_E}$$

and each of the embeddings has norm 1. Therefore any r.i. space E is intermediate for the couple $(\Lambda_{\varphi_E}, M_{\varphi_E})$.

Let φ be an arbitrary positive finite function on (a,∞), $0 \le a < \infty$, we denote its associated dilation function by

$$m_\varphi(t) = \sup_{\max\{a, \frac{a}{t}\} < s < \infty} \frac{\varphi(ts)}{\varphi(s)}, \qquad 0 < t < \infty.$$

If $m_\varphi(t)$ is finite everywhere then there exist the lower and upper extension indices of φ defined as

$$\pi_\varphi = \sup_{0<t<1} \frac{\ln m_\varphi(t)}{\ln(t)} = \lim_{t\to 0^+} \frac{\ln m_\varphi(t)}{\ln(t)}$$

and

$$\rho_\varphi = \inf_{t>1} \frac{\ln m_\varphi(t)}{\ln(t)} = \lim_{t\to\infty} \frac{\ln m_\varphi(t)}{\ln(t)}.$$

In general, $-\infty < \pi_\varphi \le \rho_\varphi < \infty$, but if φ is quasi-concave then $0 \le \pi_\varphi \le \rho_\varphi \le 1$. Note also that both indices do not change after replacing $\varphi(t)$ by arbitrary equivalent function. As an example, for the family of functions $\varphi(t) = t^\alpha(1+|\log t|)^\beta$, $\alpha, \beta \in \mathbb{R}$, the indices satisfy $\pi_\varphi = \rho_\varphi = \alpha$.

Many properties of r.i. spaces can be expressed in terms of the extension indices of their fundamental functions. For example, if $\rho_{\varphi_E} < 1$ the Hardy operator

$$(\mathcal{H}f)(t) = \frac{1}{t}\int_0^t f(s)ds$$

is bounded on an r.i. space E on $((0,\infty), dt)$. The converse is also true if we substitute ρ_{φ_E} by the upper Boyd index of E (see, e.g. [17]). Other example appears in the norm characterization of ultrasymmetric spaces defined by E. Pustylnik.

Definition 2.1 ([21])**.** Let $\varphi : (0,\infty) \longrightarrow (0,\infty)$ a function with $0 \leq \pi_\varphi \leq \rho_\varphi < \infty$. A rearrangement invariant space G is called ultrasymmetric if it is a real interpolation space between Λ_φ and M_φ. This inmediatelly implies that $\varphi_G \sim \varphi$.

These spaces satisfy the following analytical description:

Theorem 2.2 (Thm. 2.1 and 2.2. in [21])**.** *An r.i. space G with $0 < \pi_{\varphi_G} \leq \rho_{\varphi_G} < \infty$ is ultrasymmetric if and only if its norm is equivalent to*

$$\|\varphi(t)f^*(t)\|_{\widetilde{E}}$$

for some space $\widetilde{E}$ which is r.i. with respect to the measure dt/t and for any parameter function $\varphi \sim \varphi_G$. Moreover, the space E can be obtained from the couple (L_1, L_∞) by the same interpolation $\mathcal{F}$ which gives $G = \mathcal{F}(\Lambda_{\varphi_E}, M_{\varphi_E})$.

The condition $0 < \pi_{\varphi_G} \leq \rho_{\varphi_G} < \infty$ assures that the function φ_G does not grow too slowly and keeps the space G apart from L_∞. Observe that $0 = \pi_{\varphi_{L_\infty}} = \rho_{\varphi_{L_\infty}}$.

Despite the restriction on the extension indices of the fundamental function φ, last characterization allows the identification of large families of r.i. spaces as ultrasymmetric. That is the case of some Orlicz spaces, see the paper by Astashkin and Maligranda [1], Lorentz-Zygmund spaces introduced by Bennett and Rudnick in [2], generalized Lorentz-Zygmund spaces used by Edmunds, Gurka and Opic in [13] and Opic and Pick [19] or Lorentz-Zygmund type spaces studied by Cwikel and Pustylnik in [12].

In the present paper we deal with ultrasymmetric spaces that are close to L_∞ in the sense that both indices are zero. To describe the family of fundamental functions of the r.i. spaces we work with we will use the iterated logarithms:

$$\begin{cases} L_1(t) = \ell(t) = 1 + |\log t|, & t > 0 \\ L_{n+1}(t) = \ell(L_n(t)), & t > 0 \text{ and } n \in \mathbb{N}. \end{cases}$$

Similarly, for all $s \in \mathbb{R}$, we consider the iterated exponential functions

$$E_1^-(s) = e^{1-s}, \quad E_{n+1}^-(s) = E_n^-(e^{s-1}), \quad n \geq 1,$$

and

$$E_1^+(s) = e^{s-1}, \quad E_{n+1}^+(s) = E_n^+(e^{s-1}), \quad n \geq 1.$$

It is clear that, for all $n \in \mathbb{N}$, $L_n(E_n^-(t)) = t$ if $0 < t \leq 1$ and $L_n(E_n^+(t)) = t$ if $1 \leq t < \infty$.

Definition 2.3. We say that a positive function $\varphi : (0,\infty) \longrightarrow (0,\infty)$ belongs to the family $\mathcal{P}$ if it satisfies the following properties:

a) The function φ is concave or quasi-concave.
b) Both extension indices are zero, $\pi_\varphi = \rho_\varphi = 0$.
c) There exist $n_0, n_1 \in \mathbb{N}$ such that the functions Φ_0 and Φ_1 defined as

$$\Phi_0(s) = \varphi(E^-_{n_0}(s)) \quad \text{and} \quad \Phi_1(s) = \varphi(E^+_{n_1}(s)), \quad s > 1,$$

fulfil the conditions:
c.1) $\Phi_0(2t) \sim \Phi_0(t)$, $\Phi_1(2t) \sim \Phi_1(t)$, for all $t > 0$.
c.2) The dilation indices satisfy

$$\rho_{\Phi_0} < 0 < \pi_{\Phi_1}.$$

Remark 2.4. Condition c) allow us to write the function φ as

$$\varphi(t) = \begin{cases} \Phi_0(L_{n_0}(t)) & \text{if } t \in (0,1], \\ \Phi_1(L_{n_1}(t)) & \text{if } t \in (1,\infty). \end{cases} \tag{1}$$

Moreover, hypothesis c.1) on functions Φ_0 and Φ_1 implies that $\varphi(t^2) \sim \varphi(t)$, $t > 0$.

Remark 2.5. Functions in the class $\mathcal{P}$ are slowly varying functions. That is to say, if $\varphi \in \mathcal{P}$, then for every $\varepsilon > 0$ the function $t^\varepsilon \varphi(t)$, for $t > 0$, is equivalent to an increasing function while $t^{-\varepsilon}\varphi(t)$, for $t > 0$, is equivalent to a decreasing function.

Examples 2.6. Some slowly varying functions can be described in terms of $\ell(t)$ (the case $n = 1$) through functions Φ_0 and Φ_1. That is the case of the broken logarithmic functions, defined as

$$\ell^{\mathbb{A}}(t) = \begin{cases} \ell^{\alpha_0}(t), & 0 < t \le 1 \\ \ell^{\alpha_1}(t), & t > 1, \end{cases} \tag{2}$$

where $\mathbb{A} = (\alpha_0, \alpha_1) \in \mathbb{R}^2$. Observe that $\Phi_0(t) = t^{\alpha_0}$ and $\Phi_1(t) = t^{\alpha_1}$, $t > 1$, then $\ell^{\mathbb{A}} \in \mathcal{P}$ if $\alpha_0 < 0 < \alpha_1$.

However, functions with even slowlier variation may need to be described in terms of L_n, with $n > 1$. For example,

$$\mathcal{L}(t) = \begin{cases} (\ell \circ \ell)^{\alpha_1}(t) \cdot (\ell \circ \ell \circ \ell)^{\alpha_2}(t) & t \in (0,1] \\ (\ell \circ \ell \circ \ell)^{\beta}(t) & t \in (1,\infty), \end{cases}$$

can be described using $L_2(t)$ and $L_3(t)$ in the form

$$\mathcal{L}(t) = \begin{cases} L_2(t)^{\alpha_1} \cdot \ell^{\alpha_2}(L_2(t)) & t \in (0,1] \\ L_3(t)^{\beta} & t \in (1,\infty). \end{cases}$$

Hence $\mathcal{L} \in \mathcal{P}$ if α_1 and α_2 are negative and β is positive.

Next we describe the family of spaces we will use as parameters in the analytical characterization of ultrasymmetric spaces. Let $\varphi \in \mathcal{P}$ and consider the natural numbers n_0, n_1 provided by condition c) of Definition 2.3. We define the function

$$L(t) = \begin{cases} L_1(t)L_2(t)\cdots L_{n_0}(t), & t \in (0,1] \\ L_1(t)L_2(t)\cdots L_{n_1}(t), & t \in (1,\infty). \end{cases}$$

The function L depends on n_0 and n_1 which are determined by φ. Therefore L may stand for different functions if we make differents choices of φ. However, this will cause no confusion. Now we introduce the measure

$$\nu(A)=\int_0^\infty \chi_A(t)\frac{dt}{tL(t)}.$$

It is an straightforward computation to show that

$$\frac{1}{tL(t)}=\begin{cases}-\frac{L'_{n_0}(t)}{L_{n_0}(t)} & \text{if } t\in(0,1]\\ \frac{L'_{n_1}(t)}{L_{n_1}(t)} & \text{if } t\in(1,\infty)\end{cases}. \tag{3}$$

We consider the couple $(\widehat{L}_1, L_\infty)$ of Lebesgue spaces with respect to the measure ν,

$$\|f\|_{\widehat{L}_1}=\int_0^\infty |f(t)|\tfrac{dt}{tL(t)} \qquad \text{and} \qquad \|f\|_{L_\infty}=\sup_{t>0}|f(t)|.$$

Since we consider function spaces with the Fatou property, given an r.i. space E there exists an exact interpolation functor $\mathcal{F}$ that generates E from the couple (L_1, L_∞), that is to say $E=\mathcal{F}(L_1,L_\infty)$. We define $\widehat{E}$ as the space generated by the action of $\mathcal{F}$ on the couple $(\widehat{L}_1, L_\infty)$,

$$\widehat{E}=\mathcal{F}(\widehat{L}_1,L_\infty).$$

Then, $\widehat{E}$ is an r.i. space with respect to the measure ν.

The norms of the spaces E and $\widehat{E}$ can be directly connected without the use of interpolation functors. For measurable functions $f:(0,\infty)\longrightarrow(0,\infty)$ we have

$$\|f\|_{\widehat{E}(0,1)}=\|f(E^-_{n_0}(e^u))\|_E \quad \text{and} \quad \|f\|_{\widehat{E}(1,\infty)}=\|f(E^+_{n_1}(e^u))\|_E.$$

Spaces $\widehat{E}$ are often supplied with weights $\varphi:(0,\infty)\longrightarrow(0,\infty)$. We denote by $\widehat{E}^\varphi$ the space with the norm $\|f\|_{\widehat{E}^\varphi}=\|\varphi f\|_{\widehat{E}}$.

Next lemma shows basic properties of functions in the family $\mathcal{P}$.

Lemma 2.7. *Let $\varphi\in\mathcal{P}$. Then*

$$\|\varphi\|_{\widehat{L}_1(0,s)}\lesssim\varphi(s) \quad \textit{and} \quad \|1/\varphi\|_{\widehat{L}_1(s,\infty)}\lesssim 1/\varphi(s)$$

for all $s\in(0,\infty)$.

Proof. For $0<s\le 1$ we use the expression of φ on $(0,1)$ to write

$$\|\varphi\|_{\widehat{L}_1(0,s)}=\int_0^s\varphi(t)\frac{dt}{tL(t)}=\int_0^s\Phi_0(L_{n_0}(t))\frac{dt}{tL(t)}$$
$$=-\int_0^s\Phi_0(L_{n_0}(t))\frac{L'_{n_0}(t)}{L_{n_0}(t)}dt=\int_{L_{n_0}(s)}^\infty\Phi_0(u)\frac{du}{u}.$$

By hypothesis $\rho_{\Phi_0}<0$, then using Corollary 1 of [17, p. 57] we get

$$\|\varphi\|_{\widehat{L}_1(0,s)}\sim\Phi_0(L_{n_0}(s))=\varphi(s).$$

Similarly it can be proved that $\|\varphi\|_{\widehat{L}_1(0,s)}\lesssim\varphi(s)$ when $s>1$ and the second estimate. □

Now we recall the definitions of the K and J interpolation methods in the context we are going to use them. We refer to the monographs [5] and [4] for an complete account on K and J interpolation methods.

Let $\overline{A} = (A_0, A_1)$ be a Banach couple, that is, A_0 and A_1 are Banach spaces continuously embedded in some Hausdorff topological vector space. For $t > 0$, Peetres's K and J-functionals are defined by

$$\begin{aligned} K(t,a) &= K(t,a;A_0,A_1) \\ &= \inf\big\{\|a_0\|_0 + t\|a_1\|_1;\ a = a_0 + a_1,\ a_i \in A_i\big\}, \quad a \in A_0 + A_1 \end{aligned}$$

and

$$J(t,a) = J(t,a;A_0,A_1) = \max\big\{\|a\|_{A_0}, t\|a\|_{A_1}\big\}, \quad a \in A_0 \cap A_1.$$

Throughout the paper, the function

$$\widehat{\varphi}(t) = \tfrac{\varphi(t)}{t}, \quad t > 0,$$

will simplify our notation.

Definition 2.8. Let $\overline{A} = (A_0, A_1)$ be a Banach couple, $\varphi \in \mathcal{P}$ and let E be an r.i. space. The space $(A_0, A_1)^K_{\widehat{E}^{\widehat{\varphi}}}$ consists of all those elements a in $A_0 + A_1$ that satisfy

$$\|K(t,a)\|_{\widehat{E}^{\widehat{\varphi}}} < \infty.$$

These spaces can be seen as extreme K-spaces with parameter $\theta = 1$, since

$$\|a\|^K_{\widehat{E}^{\widehat{\varphi}}} = \|t^{-1}\varphi(t)K(t,a)\|_{\widehat{E}}.$$

Moreover, when $E = L_q$ for $1 \le q < \infty$, $(A_0, A_1)^K_{\widehat{E}^{\widehat{\varphi}}}$ coincides with the interpolation space $(A_0, A_1)_{1,\varphi(t)/L^{1/q}(t),L_q}$ defined in [14] by the norm

$$\|a\|^K_{\widehat{L}_q^{\widehat{\varphi}}} = \Big(\int_0^\infty \Big(\frac{K(t,a)}{tL^{1/q}(t)/\varphi(t)}\Big)^q \frac{dt}{t}\Big)^{1/q},$$

while $(A_0, A_1)^K_{\widehat{L}_\infty^{\widehat{\varphi}}} = (A_0, A_1)_{1,\varphi(t),L_\infty}$. We refer to [14], [16] and the bibliography therein, for reiteration theorems of those methods. Moreover, in [15] the spaces $(A_0, A_1)^K_{\widehat{E}^{\widehat{\varphi}}}$ are identified with extreme reiteration spaces when the function $\varphi \in \mathcal{P}$ with $L(t) = \ell(t)$, $t > 0$.

It is easy to check by standard arguments that the space $(A_0, A_1)^K_{\widehat{E}^{\widehat{\varphi}}}$ is not only a Banach space but also an *interpolation space* for the couple (A_0, A_1). See [4] or [14, Prop. 3.2].

Next we introduce J interpolation spaces.

Definition 2.9. Let $\overline{A} = (A_0, A_1)$ be a Banach couple, $\varphi \in \mathcal{P}$ and let E be an r.i. space. We say an element $a \in A_0 + A_1$ belongs to the space $(A_0, A_1)^J_{\widehat{E}^{\widehat{\varphi}L}}$ if there exists a representation of a as

$$a = \int_0^\infty u(t)\frac{dt}{t} \tag{4}$$

where u is a strong measurable function with values in $A_0 \cap A_1$, satisfying that

$$\big\|J(t,u(t))\big\|_{\widehat{E}^{\widehat{\varphi}L}} < \infty. \tag{5}$$

The norm of the element a in the space $(A_0, A_1)^J_{\widehat{E}^{\widehat{\varphi}L}}$ is given by

$$\|a\|^J_{\widehat{E}^{\widehat{\varphi}L}} = \inf \Big\{ \|J(t, u(t))\|_{\widehat{E}^{\widehat{\varphi}L}} \Big\},$$

where the infimum is taken over all representations of a satisfying (4) and (5).

Standard arguments show that

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)^J_{\widehat{E}^{\widehat{\varphi}L}}.$$

In the next section, we will prove that the space $(A_0, A_1)^J_{\widehat{E}^{\widehat{\varphi}L}}$ is embedded in the sum $A_0 + A_1$, establishing that it is an intermediate space for the couple (A_0, A_1). Actually, it is an interpolation space too. See [4], § 3.2.

3. J-K Identities

Now we obtain an equivalence theorem between the K and J limiting interpolation spaces and a reiteration result. Let us begin with the embedding

$$\big(A_0, A_1\big)^J_{\widehat{E}^{\widehat{\varphi}L}} \hookrightarrow \big(A_0, A_1\big)^K_{\widehat{E}^{\widehat{\varphi}}}.$$

J and K-functionals of any element a in $A_0 \cap A_1$ are related by the inequality

$$K(t, a) \leq \min\big\{1, \tfrac{t}{s}\big\} J(s, a), \quad t, s > 0.$$

This makes the operator

$$\big(Sf\big)(t) = \int_0^\infty \min\{1, \tfrac{t}{s}\} f(s) \tfrac{ds}{s}$$

an important tool to establish embeddings of J-spaces into the K-spaces. Next result will be useful for this purpose.

Proposition 3.1. *Let $\varphi \in \mathcal{P}$ and let E be an r.i space. Then, the operator*

$$S : \widehat{E}^{\widehat{\varphi}L} \longrightarrow \widehat{E}^{\widehat{\varphi}} \tag{6}$$

is bounded.

Proof. We prove first that the operator S is bounded between L_1 spaces. Let $f \in \widehat{L}_1^{\widehat{\varphi}L}$, then

$$\begin{aligned}
\|Sf(t)\|_{\widehat{L}_1^{\widehat{\varphi}}} &\leq \int_0^\infty \widehat{\varphi}(t) \int_0^\infty \min\{1, \tfrac{t}{s}\} |f(s)| \tfrac{ds}{s} \tfrac{dt}{tL(t)} \\
&= \int_0^\infty \frac{\varphi(t)}{t} \Big(\int_0^t |f(s)| \tfrac{ds}{s} + \int_t^\infty \tfrac{t}{s} |f(s)| \tfrac{ds}{s} \Big) \tfrac{dt}{tL(t)} \\
&= \int_0^\infty \Big(\int_s^\infty \frac{\varphi(t)}{tL(t)} \tfrac{dt}{t} \Big) |f(s)| \tfrac{ds}{s} + \int_0^\infty \bigg(\int_0^s \varphi(t) \tfrac{dt}{tL(t)} \bigg) \tfrac{|f(s)|}{s} \tfrac{ds}{s}
\end{aligned}$$

By Lemma II.1.5 of [17] and Lemma 2.7, we get

$$
\begin{aligned}
\|Sf\|_{\widehat{L}_1^{\widehat{\varphi}}} &\lesssim \int_0^\infty \tfrac{\varphi(s)}{sL(s)}|f(s)|\tfrac{ds}{s} + \int_0^\infty \frac{\varphi(s)}{s}|f(s)|\tfrac{ds}{s} \\
&= \int_0^\infty \widehat{\varphi}(s)(\tfrac{1}{L(s)}+1)|f(s)|\tfrac{ds}{s} \\
&\lesssim \|f\|_{\widehat{L}_1^{\widehat{\varphi}L}}.
\end{aligned}
$$

Now we proceed to prove that the operator $S : \widehat{L}_\infty^{\widehat{\varphi}L} \longrightarrow \widehat{L}_\infty^{\widehat{\varphi}}$ is also bounded. We claim that

$$
\int_0^\infty \min\left\{1, \tfrac{t}{s}\right\} \tfrac{1}{\widehat{\varphi}(s)L(s)}\tfrac{ds}{s} \lesssim \tfrac{1}{\widehat{\varphi}(t)}\left(\tfrac{1}{L(t)}+1\right), \quad t>0. \tag{7}
$$

Actually, by Lemma II.1.4 of [17]

$$
\int_0^t \tfrac{1}{\widehat{\varphi}(s)L(s)}\tfrac{ds}{s} = \int_0^t \tfrac{s}{\varphi(s)L(s)}\tfrac{ds}{s} \sim \tfrac{t}{\varphi(t)L(t)} = \tfrac{1}{\widehat{\varphi}(t)L(t)}, \quad t>0.
$$

On the other hand, Lemma 2.7 assures that

$$
\int_t^\infty \tfrac{1}{s\widehat{\varphi}(s)L(s)}\tfrac{ds}{s} \lesssim \tfrac{1}{t\widehat{\varphi}(t)}, \quad t>0.
$$

This establishes (7). Now choose $f \in \widehat{L}_\infty^{\widehat{\varphi}L}$, then

$$
\begin{aligned}
\|Sf\|_{\widehat{L}_\infty^{\widehat{\varphi}}} &\le \sup_{0<t<\infty} \widehat{\varphi}(t)\int_0^\infty \min\{1, \tfrac{t}{s}\}|f(s)|\frac{ds}{s} \\
&\le \|f\|_{\widehat{L}_\infty^{\widehat{\varphi}L}} \sup_{0<t<\infty} \widehat{\varphi}(t)\int_0^\infty \min\left\{1, \tfrac{t}{s}\right\}\frac{1}{\widehat{\varphi}(s)L(s)}\frac{ds}{s} \\
&\lesssim \|f\|_{\widehat{L}_\infty^{\widehat{\varphi}L}}.
\end{aligned}
$$

Finally, by interpolation we get that the operator (6) is bounded. □

We are now in position to establish the embedding of the J-space into the K-space.

Theorem 3.2. *Let $\overline{A} = (A_0, A_1)$ be a Banach couple and let $\varphi \in \mathcal{P}$, then*

$$
\left(A_0, A_1\right)^J_{\widehat{E}^{\widehat{\varphi}L}} \hookrightarrow \left(A_0, A_1\right)^K_{\widehat{E}^{\widehat{\varphi}}}.
$$

Proof. Let $a = \int_0^\infty u(t)\frac{dt}{t}$ be a representation of $a \in \left(A_0, A_1\right)^J_{\widehat{E}^{\widehat{\varphi}L}}$ such that

$$
\|J(t, u(t))\|_{\widehat{E}^{\widehat{\varphi}L}} \le (1+\varepsilon)\|a\|^J_{\widehat{E}^{\widehat{\varphi}L}}
$$

for some $\varepsilon > 0$. The K-functional of a can be estimated in terms of the operator S acting on the J-functional as follows. For $t, s > 0$,

$$
\begin{aligned}
K(t,a) = K\left(t, \int_0^\infty u(s)\frac{ds}{s}\right) &\le \int_0^\infty K(t, u(s))\frac{ds}{s} \\
&\le \int_0^\infty \min\{1, \tfrac{t}{s}\}J(s, u(s))\frac{ds}{s} = S\big(J(s, u(s))\big)(t).
\end{aligned}
$$

Therefore, by Proposition 3.1, we have

$$\begin{aligned}\|K(t,a)\|_{\widehat{E}\widehat{\varphi}} &\le \|SJ(t,u(t))\|_{\widehat{E}\widehat{\varphi}} \le \|S\|\,\|J(t,u(t))\|_{\widehat{E}\widehat{\varphi}L}\\ &\le (1+\varepsilon)\|S\|\,\|a\|^{J}_{\widehat{E}\widehat{\varphi}L},\end{aligned}$$

where $\|S\|$ stands for the norm of the operator (6). This completes the proof. $\square$

As a direct consequence we get that the J-space is embedded in the sum, so it is intermediate for the couple (A_0, A_1).

Now we focus on the reverse embedding

$$\big(A_0, A_1\big)^{K}_{\widehat{E}\widehat{\varphi}} \hookrightarrow \big(A_0, A_1\big)^{J}_{\widehat{E}\widehat{\varphi}L}.$$

The following properties of the K-functional for the elements in the space $(A_0, A_1)^{K}_{\widehat{E}\widehat{\varphi}}$ play an important role in the process of showing that every element in $(A_0, A_1)^{K}_{\widehat{E}\widehat{\varphi}}$ is also an element of the space $(A_0, A_1)^{J}_{\widehat{E}\widehat{\varphi}L}$.

Proposition 3.3. *Let $\overline{A} = (A_0, A_1)$ be a Banach couple and let $\varphi \in \mathcal{P}$. If $a \in (A_0, A_1)^{K}_{\widehat{E}\widehat{\varphi}}$, then*

$$\begin{aligned}K(t,a) &\to 0 \qquad \textit{as } t \to 0,\\ \frac{K(t,a)}{t} &\to 0 \qquad \textit{as } t \to \infty.\end{aligned}$$

Proof. Choose $s < 1$ such that $L_{n_0}(s) > e$ and let $b = E^{-}_{n_0}(\frac{1}{e}L_{n_0}(s))$. Then, using the concavity of the K-functional and of φ, we get

$$\begin{aligned}\|a\|_{\overline{A}^{K}_{\widehat{E}\widehat{\varphi}}} &= \|\widehat{\varphi}(t)K(t,a)\|_{\widehat{E}} \ge \|\tfrac{\varphi(t)}{t}K(t,a)\|_{\widehat{E}(s,b)}\\ &\ge K(s,a)\tfrac{\varphi(b)}{b}\|\chi_{(s,b)}\|_{\widehat{E}} = K(s,a)\tfrac{\varphi(b)}{b}\varphi_{\widehat{E}}(1).\end{aligned}$$

Hence, $\frac{b}{\varphi(b)}\|a\|_{\overline{A}^{K}_{\widehat{E}\widehat{\varphi}}} \gtrsim K(s,a) \ge 0$. Now, if $s \to 0$ then $b \to 0$ and $\frac{b}{\varphi(b)} \to 0$ (since the lower index of the function $\frac{t}{\varphi(t)}$ is strictly positive) which proves that $K(s,a) \to 0$ if $s \to 0$.

For the second limit choose $s > 1$ such that $L_{n_1}(s) > e$ and let $b = E^{+}_{n_1}(\frac{1}{e}L_{n_1}(s))$. Then

$$\begin{aligned}\|a\|_{\overline{A}^{K}_{\widehat{E}\widehat{\varphi}}} &= \|\widehat{\varphi}(t)K(t,a)\|_{\widehat{E}} \ge \|\varphi(t)\tfrac{K(t,a)}{t}\|_{\widehat{E}(b,s)}\\ &\ge \tfrac{K(s,a)}{s}\varphi(b)\|\chi_{(b,s)}\|_{\widehat{E}} = \tfrac{K(s,a)}{s}\varphi(b)\varphi_{\widehat{E}}(1).\end{aligned}$$

Since $\pi_{\Phi_1} > 0$, $\lim_{s\to\infty}\varphi(E^{+}_{n_1}(\frac{1}{e}L_{n_1}(s)) = \lim_{s\to\infty}\Phi_1(\frac{1}{e}L_{n_1}(s)) = \infty$ and we deduce that $\lim_{s\to\infty}\frac{K(s,a)}{s} = 0$. $\square$

Next result gives the inclusion of the K-space into the J-space.

Theorem 3.4. *Let $\varphi \in \mathcal{P}$ and E be a r.i. space. Then, for any Banach couple $\overline{A} = (A_0, A_1)$,*

$$(A_0, A_1)^{K}_{\widehat{E}\widehat{\varphi}} \hookrightarrow (A_0, A_1)^{J}_{\widehat{E}\widehat{\varphi}L}.$$

Proof. Let us begin by establishing a partition of $(0,\infty)$. Define the increasing sequence $(\lambda_\nu)_{\nu\in\mathbb{Z}}$ as

$$\lambda_\nu = \begin{cases} E^-_{n_0}(2^{-\nu}) & \text{if } -\nu \in \mathbb{N}, \\ E^+_{n_1}(2^{\nu}) & \text{if } \nu \in \mathbb{N}\cup\{0\}, \end{cases}$$

and the family of intervals

$$I_\nu = \begin{cases} (\lambda_{\nu-1}, \lambda_\nu] & \text{if } -\nu \in \mathbb{N} \\ (\lambda_{-1}, \lambda_0) & \text{if } \nu = 0 \\ [\lambda_{\nu-1}, \lambda_\nu) & \text{if } \nu \in \mathbb{N}. \end{cases}$$

Let $a \in (A_0, A_1)^K_{\widehat{E}\widehat{\varphi}}$. Using the properties of the K-functional, we can find decompositions $a = a_{1,\nu} + a_{0,\nu}$, $a_{i,\nu} \in A_i$, such that

$$\|a_{0,\nu}\|_{A_0} + \lambda_{\nu-1}\|a_{1,\nu}\|_{A_1} \leq 2K(\lambda_{\nu-1}, a) \quad \text{ for } \nu \in \mathbb{Z}.$$

The elements $v_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu}$ in $A_0 \cap A_1$ represent the element a through the series $a = \sum_{\nu\in\mathbb{Z}} v_\nu$. Actually

$$\begin{aligned} a - \sum_{\nu=-N}^{M} v_\nu &= a - \sum_{\nu=-N}^{M} (a_{0,\nu} - a_{0,\nu-1}) = a - (a_{0,M} - a_{0,-N-1}) \\ &= a_{1,M} - a_{0,-N-1}. \end{aligned}$$

Moreover, by Lemma 3.3,

$$\begin{aligned} \Big\| a - \sum_{\nu=-N}^{M} v_\nu \Big\|_{A_0+A_1} &\leq \|a_{0,-N-1}\|_{A_0} + \|a_{1,M}\|_{A_1} \\ &\leq 2K(\lambda_{-N-2}, a) + 2\frac{1}{\lambda_{M-1}} K(\lambda_{M-1}, a) \to 0 \end{aligned}$$

as $M, N \to \infty$ since $\lambda_{-N-2} \to 0$ as $N \to \infty$ and $\lambda_{M-1} \to \infty$ as $M \to \infty$.

In order to prove that a is an element of the J-space an integral representation of a is required, so we define

$$v(t) = \frac{v_\nu}{\mu(I_\nu)L(t)} \quad \text{ if } t \in I_\nu \text{ with } \nu \in \mathbb{Z},$$

where $\mu(I_\nu) = \int_{I_\nu} \frac{dt}{tL(t)} = \log(2)$. Clearly

$$\int_0^\infty v(t)\frac{dt}{t} = \sum_{\nu\in\mathbb{Z}} v_\nu = a.$$

Next we establish estimates for the function $L(t)J(t, v(t), A_0, A_1)/t$, $t > 0$, on every interval I_ν, $\nu \in \mathbb{Z}$. Take $t \in I_\nu$ with $\nu \in \mathbb{Z}$, using the monotonicity of the

K-functional and of the sequence $(\lambda_\nu)_{\nu\in\mathbb{Z}}$, we have

$$\begin{aligned}
\frac{L(t)}{t}J(t,v(t)) &\lesssim \max\{\frac{1}{t}\|v_\nu\|_{A_0}, \|v_\nu\|_{A_1}\} \\
&\le \max\left\{\frac{1}{\lambda_{\nu-1}}\|a_{0,\nu}-a_{0,\nu-1}\|_{A_0}, \|a_{1,\nu-1}-a_{1,\nu}\|_{A_1}\right\} \\
&\le \left[\frac{1}{\lambda_{\nu-1}}\|a_{0,\nu}\|_{A_0} + \|a_{1,\nu}\|_{A_1} + \frac{1}{\lambda_{\nu-1}}\|a_{0,\nu-1}\|_{A_0} + \|a_{1,\nu-1}\|_{A_1}\right] \\
&\lesssim \frac{1}{\lambda_{\nu-2}}K(\lambda_{\nu-2},a).
\end{aligned}$$

Since E is a rerrangement-invariant space, using that $\bigcup_{\nu\in\mathbb{Z}} I_\nu = (0,\infty)$ and the previous estimates, we derive

$$\begin{aligned}
\|a\|^J_{\widehat{E}^{\varphi L}} &\le \left\|\frac{\varphi(t)}{t}L(t)J(t,v(t))\right\|_{\widehat{E}} \\
&= \left\|\sum_{\nu\in\mathbb{Z}}\frac{\varphi(t)}{t}L(t)J(t,v(t))\chi_{I_\nu}(t)\right\|_{\widehat{E}} \\
&\lesssim \left\|\sum_{\nu\le -2}\varphi(t)\frac{1}{\lambda_{\nu-2}}K(\lambda_{\nu-2},a)\chi_{I_\nu}(t)\right\|_{\widehat{E}} \qquad (8)\\
&+ \left\|\sum_{\nu\ge 3}\varphi(t)\frac{1}{\lambda_{\nu-2}}K(\lambda_{\nu-2},a)\chi_{I_\nu}(t)\right\|_{\widehat{E}} \qquad (9)\\
&+ \left\|\sum_{\nu=-1}^{2}\varphi(t)K(\lambda_{\nu-2},a)\chi_{I_\nu}(t)\right\|_{\widehat{E}}. \qquad (10)
\end{aligned}$$

In order to estimate summand (8) we notice that if $t\in I_\nu$ with $\nu\le -2$ then $E^-_{n_0}(4L_{n_0}(t))\in I_{\nu-2}=(\lambda_{\nu-3},\lambda_{\nu-2}]$ and so

$$\frac{1}{\lambda_{\nu-2}}K(\lambda_{\nu-2},a)\le \frac{1}{E^-_{n_0}(4L_{n_0}(t))}K(E^-_{n_0}(4L_{n_0}(t)),a).$$

Moreover, $\varphi(t)=\Phi_0(L_{n_0}(t))\sim\Phi_0(4L_{n_0}(t)))=\varphi(E^-_{n_0}(4L_{n_0}(t)))$ for $t\in I_\nu$ with $\nu\le -2$. Then

$$\begin{aligned}
&\left\|\sum_{\nu\le -2}\frac{\varphi(t)}{\lambda_{\nu-2}}K(\lambda_{\nu-2},a)\chi_{I_\nu}(t)\right\|_{\widehat{E}} \\
&\qquad\lesssim \left\|\sum_{\nu\le -2}\frac{\varphi(E^-_{n_0}(4L_{n_0}(t)))}{E^-_{n_0}(4L_{n_0}(t))}K(E^-_{n_0}(4L_{n_0}(t)),a)\chi_{I_\nu}(t)\right\|_{\widehat{E}} \\
&\qquad= \left\|\frac{\varphi(E^-_{n_0}(4L_{n_0}(t)))}{E^-_{n_0}(4L_{n_0}(t))}K(E^-_{n_0}(4L_{n_0}(t)),a)\right\|_{\widehat{E}(0,E^-_{n_0}(4))} \\
&\qquad= \left\|\frac{\varphi(t)}{t}K(t,a)\right\|_{\widehat{E}(0,1)}.
\end{aligned}$$

Last estimate is due to the fact that

$$\begin{array}{rccc} \sigma^- : & \Big((0,1), \dfrac{dt}{tL(t)}\Big) & \longrightarrow & \Big((0, E^-_{n_0}(4)), \dfrac{dt}{tL(t)}\Big) \\ & t & \longmapsto & E^-_{n_0}(4L_{n_0}(t)) \end{array}$$

is a measure preserving transformation, so $f \circ \sigma^-$ and f are equimesurable and

$$\|f \circ \sigma^-\|_{\widehat{E}(0,E^-_{n_0}(4))} = \|f\|_{\widehat{E}(0,1)}.$$

Similarly, to estimate the term (9) notice that if $t \in I_\nu$ with $\nu \geq 3$, then $E^+_{n_1}(\frac{1}{4}L_{n_1}(t)) \in I_{\nu-2} = [\lambda_{\nu-3}, \lambda_{\nu-2})$, therefore

$$\frac{1}{\lambda_{\nu-2}} K(\lambda_{\nu-2}, a) \leq \frac{1}{E^+_{n_1}(\frac{1}{4}L_{n_1}(t))} K(E^+_{n_1}(\tfrac{1}{4}L_{n_1}(t)), a).$$

Now $\varphi(t) \sim \varphi(E^+_{n_1}(\frac{1}{4}L_{n_1}(t)))$ for $t \in I_\nu$ with $\nu \geq 3$, and the transformation

$$\begin{array}{rccc} \sigma^+ : & \Big((1,\infty), \dfrac{dt}{tL(t)}\Big) & \longrightarrow & \Big((E^+_{n_1}(4), \infty), \dfrac{dt}{tL(t)}\Big) \\ & t & \longmapsto & E^+_{n_1}(\frac{1}{4}L_{n_1}(t)) \end{array}$$

preserves the measure. Using the same techniques as above we obtain the estimates

$$\begin{aligned} &\Big\| \sum_{\nu \geq 3} \frac{\varphi(t)}{\lambda_{\nu-2}} K(\lambda_{\nu-2}, a) \chi_{I_\nu}(t) \Big\|_{\widehat{E}} \\ &\qquad \lesssim \Big\| \sum_{\nu \geq 3} \frac{\varphi(E^+_{n_1}(\frac{1}{4}L_{n_1}(t)))}{E^+_{n_1}(\frac{1}{4}L_{n_1}(t))} K(E^+_{n_1}(\tfrac{1}{4}L_{n_1}(t)), a) \chi_{I_\nu}(t) \Big\|_{\widehat{E}} \\ &\qquad \leq \Big\| \frac{\varphi(E^+_{n_1}(\frac{1}{4}L_{n_1}(t)))}{E^+_{n_1}(\frac{1}{4}L_{n_1}(t))} K(E^+_{n_1}(\tfrac{1}{4}L_{n_1}(t)), a) \Big\|_{\widehat{E}(E^+_{n_1}(4),\infty)} \\ &\qquad \sim \Big\| \frac{\varphi(t)}{t} K(t,a) \Big\|_{\widehat{E}(1,\infty)}. \end{aligned}$$

The remaining term, (10), obviously satisfies the inequality

$$\Big\| \sum_{\nu=-1}^{2} \varphi(t) K(\lambda_{\nu-2}, a) \chi_{I_\nu}(t) \Big\|_{\widehat{E}} \lesssim \Big\| \frac{\varphi(t)}{t} K(t,a) \Big\|_{\widehat{E}}.$$

Putting together the previous estimates

$$\|a\|^J_{\widehat{E}\widehat{\varphi}L} \lesssim \Big\| \frac{\varphi(t)}{t} K(t,a) \Big\|_{\widehat{E}} = \|a\|^K_{\widehat{E}\widehat{\varphi}}$$

and the theorem is proved. □

Theorems 3.2 and 3.4 yield the equivalence theorem.

Theorem 3.5. *Let $\overline{A} = (A_0, A_1)$ be a Banach couple. Then, for any $\varphi \in \mathcal{P}$ and any r.i. space E*

$$(A_0, A_1)^J_{\widehat{E}\widehat{\varphi}L} = (A_0, A_1)^K_{\widehat{E}\widehat{\varphi}}.$$

Now we are in a position to establish the reiteration theorem. The proof follows classical techniques (see [5]).

Theorem 3.6. *Let $\mathcal{F}$ be an interpolation functor and $\overline{A} = (A_0, A_0)$ a Banach couple. Then, for any $\varphi \in \mathcal{P}$*

$$\mathcal{F}\Big((A_0, A_1)^K_{\widehat{L}_1^{\varphi}}, (A_0, A_1)^K_{\widehat{L}_\infty^{\varphi}}\Big) = (A_0, A_1)^K_{\widehat{E}^{\varphi}} \tag{11}$$

where $\widehat{E} = \mathcal{F}\big(\widehat{L}_1^{\varphi}, \widehat{L}_\infty^{\varphi}\big)$.

Proof. Consider the couple $\overline{L}_1 = \big(L_1, L_1(\frac{1}{t})\big)$. The J interpolation method $(\cdot\,,\cdot)^J_{\widehat{E}^{\varphi L}}$ is minimal among all those methods $\mathcal{M}$ that satisfy the embedding

$$\widehat{E}^{\varphi L} \hookrightarrow \mathcal{M}(\overline{L}_1).$$

Let us consider the interpolation method

$$\mathcal{F}\Big((\cdot,\cdot)^J_{\widehat{L}_1^{\varphi L}}, (\cdot,\cdot)^J_{\widehat{L}_\infty^{\varphi L}}\Big).$$

If we make it act on the couple $\overline{L}_1$ we get

$$\mathcal{F}\Big((\overline{L}_1)^J_{\widehat{L}_1^{\varphi L}}, (\overline{L}_1)^J_{\widehat{L}_\infty^{\varphi L}}\Big) = \mathcal{F}\Big(\widehat{L}_1^{\varphi L}, \widehat{L}_\infty^{\varphi L}\Big) = \widehat{E}^{\varphi L}.$$

Therefore, by the minimality property of the J-method

$$(A_0, A_1)^J_{\widehat{E}^{\varphi L}} \hookrightarrow \mathcal{F}\Big((A_0, A_1)^J_{\widehat{L}_1^{\varphi L}}, (A_0, A_1)^J_{\widehat{L}_\infty^{\varphi L}}\Big). \tag{12}$$

Similarly, consider the couple $\overline{L}_\infty = \big(L_\infty, L_\infty(\frac{1}{t})\big)$ and recall that the K-interpolation method $(\cdot\,,\cdot)^K_{\widehat{E}^{\varphi}}$ is maximal among all those methods $\mathcal{M}$ that acting on the couple $\overline{L}_\infty$ satisfy that

$$\mathcal{M}(\overline{L}_\infty) \hookrightarrow \widehat{E}^{\varphi}.$$

In particular, if we choose the interpolation method

$$\mathcal{F}\Big((\cdot,\cdot)^K_{\widehat{L}_1^{\varphi}}, (\cdot,\cdot)^K_{\widehat{L}_\infty^{\varphi}}\Big)$$

and we make it act on the couple $\overline{L}_\infty$ we obtain

$$\mathcal{F}\Big((\overline{L}_\infty)^K_{\widehat{L}_1^{\varphi}}, (\overline{L}_\infty)^K_{\widehat{L}_\infty^{\varphi}}\Big) = \mathcal{F}\Big(\widehat{L}_1^{\varphi}, \widehat{L}_\infty^{\varphi}\Big) = \widehat{E}^{\varphi}.$$

Therefore, the maximality of the K-method acting on the couple $\overline{L}_\infty$ yields

$$\mathcal{F}\Big((A_0, A_1)^K_{\widehat{L}_1^{\varphi}}, (A_0, A_1)^K_{\widehat{L}_\infty^{\varphi}}\Big) \hookrightarrow (A_0, A_1)^K_{\widehat{E}^{\varphi}}. \tag{13}$$

Now, the combination of embeddings (12), (13) and Theorem 3.2 establishes

$$\overline{A}^J_{\widehat{E}^{\varphi L}} \hookrightarrow \mathcal{F}\Big(\overline{A}^J_{\widehat{L}_1^{\varphi L}}, \overline{A}^J_{\widehat{L}_\infty^{\varphi L}}\Big) \hookrightarrow \mathcal{F}\Big(\overline{A}^K_{\widehat{L}_1^{\varphi}}, \overline{A}^K_{\widehat{L}_\infty^{\varphi}}\Big) \hookrightarrow \overline{A}^K_{\widehat{E}^{\varphi}},$$

which together with Theorem 3.4 gives the equality

$$\mathcal{F}\Big((A_0, A_1)^K_{\widehat{L}_1^{\varphi}}, (A_0, A_1)^K_{\widehat{L}_\infty^{\varphi}}\Big) = (A_0, A_1)^K_{\widehat{E}^{\varphi}}.$$

This completes the proof. □

4. Ultrasymmetric spaces

In this section we provide the analytical characterization of ultrasymmetric spaces whose fundamental functions lie in the class $\mathcal{P}$.

We start describing Lorentz and Marcinkiewicz spaces, with fundamental function $\varphi \in \mathcal{P}$, as K interpolation spaces for the couple (L_1, L_∞). For this purpose, we will need the following assertion.

Lemma 4.1. *Let $\varphi : (0,\infty) \longrightarrow (0,\infty)$ be a function with upper index $\rho_\varphi < 1$. Then, for any r.i. space E, the Hardy operator*

$$(\mathcal{H}f)(t) = \frac{1}{t}\int_0^t f(s)ds, \qquad t > 0, \tag{14}$$

is bounded on the space $\widehat{E}^\varphi$. In particular,

$$\|\varphi(t)f^{**}(t)\|_{\widehat{E}} \sim \|\varphi(t)f^{*}(t)\|_{\widehat{E}}. \tag{15}$$

Proof. First we prove that the Hardy operator (14) is bounded on each of the spaces $\widehat{L}_1^\varphi$ and $\widehat{L}_\infty^\varphi$. Actually, let $f \in \widehat{L}_1^\varphi$, then

$$\begin{aligned}
\|\mathcal{H}f\|_{\widehat{L}_1^\varphi} &\le \int_0^\infty \varphi(t)\frac{1}{t}\int_0^t |f(s)|ds\,\frac{dt}{tL(t)} \\
&= \int_0^\infty |f(s)| \int_s^\infty \frac{\varphi(t)}{tL(t)}\frac{dt}{t}\,ds.
\end{aligned}$$

The condition $\rho_\varphi < 1$ ensures that the function $\varphi(t)/tL(t)$, $t > 0$, has upper index less than zero. So $\int_s^\infty \varphi(t)/tL(t)\frac{dt}{t} \sim \varphi(s)/sL(s)$ (see [17, p. 57]) and

$$\|\mathcal{H}f\|_{\widehat{L}_1^\varphi} \lesssim \int_0^\infty \frac{\varphi(s)}{sL(s)}|f(s)|ds = \|f\|_{\widehat{L}_1^\varphi}.$$

The norm of operator $\mathcal{H} : \widehat{L}_\infty^\varphi \longrightarrow \widehat{L}_\infty^\varphi$ can be estimated similarly,

$$\begin{aligned}
\|\mathcal{H}f\|_{\widehat{L}_\infty^\varphi} &\le \sup_{0<t<\infty} \varphi(t)\frac{1}{t}\int_0^t |f(s)|ds \\
&= \sup_{0<t<\infty} \frac{\varphi(t)}{t}\int_0^t \varphi(s)|f(s)|\frac{s}{\varphi(s)}\frac{ds}{s} \\
&\le \|f\|_{\widehat{L}_\infty^\varphi}\frac{\varphi(t)}{t}\int_0^t \frac{s}{\varphi(s)}\frac{ds}{s} \\
&\lesssim \|f\|_{\widehat{L}_\infty^\varphi}.
\end{aligned}$$

Last equivalence follows from the fact that the function $t/\varphi(t)$ has strictly positive lower index (see [17, p. 57]).

Now, since $\widehat{E}^\varphi$ is an interpolation space between $(\widehat{L}_1^\varphi, \widehat{L}_\infty^\varphi)$, we obtain that the Hardy operator (14) is bounded on the space $\widehat{E}^\varphi$. In particular, for any $f \in \widehat{E}^\varphi$

$$\|\varphi(t)f^{**}(t)\|_{\widehat{E}} \lesssim \|\varphi(t)f^{*}(t)\|_{\widehat{E}}.$$

This, together with the fact that $f^*(t) \le f^{**}(t)$ for $t > 0$, yields the equivalence (15). □

Proposition 4.2. *Let φ be a function in $\mathcal{P}$, and consider the Lorentz and Marcinkiewicz spaces associated to φ, Λ_φ and M_φ. Then*

$$\Lambda_\varphi = (L_1, L_\infty)^K_{\widehat{L}_1^{\widehat{\varphi}}} \quad \textit{and} \quad M_\varphi = (L_1, L_\infty)^K_{\widehat{L}_\infty^{\widehat{\varphi}}}.$$

Proof. Let L_{n_i} and Φ_i, $i = 0, 1$, be the functions that appear in the description (1) of φ. By hypothesis the dilation indices of Φ_0 and Φ_1 satisfy $\rho_{\Phi_0} < 0 < \pi_{\Phi_1}$, then there exist equivalent differentiable functions, which we denote in the same way, such that

$$\frac{t\Phi_0'(t)}{\Phi_0(t)} \sim -1 \quad \text{and} \quad \frac{t\Phi_1'(t)}{\Phi_1(t)} \sim 1,$$

for all $t \geq 1$ (see [24, Lemma 2.1]). Hence we can assume that φ is a differentiable function with

$$\varphi'(t) = \begin{cases} \Phi_0'(L_{n_0}(t))L_{n_0}'(t) & t \in (0,1) \\ \Phi_1'(L_{n_1}(t))L_{n_1}'(t) & t \in (1,\infty) \end{cases}.$$

Using (3) we get that $\varphi'(t) \sim \varphi(t)\frac{1}{tL(t)}$ for all $t > 0$.

Now, using Lemma 4.1 and the above equivalences, we have that for any function $f \in \Lambda_\varphi$

$$\begin{aligned} \|f\|_{\Lambda_\varphi} &= \varphi(0^+)\|f\|_{L_\infty} + \int_0^\infty f^*(t)d\varphi(t) = \int_0^\infty f^*(t)\varphi'(t)dt \\ &\sim \int_0^\infty f^*(t)\varphi(t)\frac{dt}{tL(t)} \sim \int_0^\infty f^{**}(t)\varphi(t)\frac{dt}{tL(t)} \\ &= \int_0^\infty K(t,f;L_1,L_\infty)\frac{\varphi(t)}{t}\frac{dt}{tL(t)} \\ &= \|f\|_{(L_1,L_\infty)^K_{\widehat{L}_1^{\widehat{\varphi}}}}. \end{aligned}$$

Notice that $\varphi(0^+) = 0$ since $\pi_{\Phi_0} < 0$ and Petree' s result $K(t,f;L_1,L_\infty) = tf^{**}(t)$, $t > 0$.

Similarly, having in mind that $L_\infty = \widehat{L}_\infty$ and Lemma 4.1, any $f \in M_\varphi$ satisfies

$$\|f\|_{M_\varphi} = \|\varphi(t)f^{**}(t)\|_{L_\infty} = \|\widehat{\varphi}(t)K(t,f;L_1,L_\infty)\|_{\widehat{L}_\infty} = (L_1,L_\infty)^K_{\widehat{L}_\infty^{\widehat{\varphi}}}.$$

□

The previous result can be compared with the classical result that assures that if the parameter function φ has both dilation indices strictly between 0 and 1 then

$$\Lambda_\varphi = (L_1, L_\infty)^K_{\widetilde{L}_1^{\widehat{\varphi}}} \quad \text{and} \quad M_\varphi = (L_1, L_\infty)^K_{L_\infty^{\widehat{\varphi}}},$$

where $\widetilde{L}_1$ is the space L_1 with the homogeneous measure dt/t and, as usual, $\widehat{\varphi}(t) = \varphi(t)/t$, $t > 0$.

Now we are in a position to establish our main result.

Theorem 4.3. *An r.i. space G, with fundamental function $\varphi_G \in \mathcal{P}$, is ultrasymmetric if and only if its norm is equivalent to*

$$\|f\|_G \sim \|\varphi(t)f^*(t)\|_{\widehat{E}}$$

for some space $\widehat{E}$ which is an r.i. space with respect to the measure $dt/tL(t)$ and for any parameter function $\varphi \sim \varphi_G$. Moreover, if $\mathcal{F}$ is the interpolation method that generates G from the couple $(\Lambda_\varphi, M_\varphi)$ then $\widehat{E} = \mathcal{F}\big(\widehat{L}_1, \widehat{L}_\infty\big)$.

Proof. Take $\varphi = \varphi_G$ and let $\mathcal{F}$ be the interpolation method that generates the ultrasymmetric space G as

$$G = \mathcal{F}(\Lambda_\varphi, M_\varphi).$$

Then, by Proposition 4.2 and Theorem 3.6 we get

$$G = \mathcal{F}(\Lambda_\varphi, M_\varphi) = \mathcal{F}\Big(\big(L_1, L_\infty\big)^K_{\widehat{L}_1^{\widehat{\varphi}}}, \big(L_1, L_\infty\big)^K_{\widehat{L}_\infty^{\widehat{\varphi}}}\Big) = \big(L_1, L_\infty\big)^K_{\widehat{E}^{\widehat{\varphi}}},$$

where $\widehat{E} = \mathcal{F}(\widehat{L}_1, \widehat{L}_\infty)$. Hence

$$\begin{aligned}\|f\|_G = \|K(t, f; L_1, L_\infty)\|_{\widehat{E}^{\widehat{\varphi}}} = \|tf^{**}(t)\|_{\widehat{E}^{\widehat{\varphi}}} = \|\varphi(t)f^{**}(t)\|_{\widehat{E}} \\ \sim \|\varphi(t)f^{*}(t)\|_{\widehat{E}},\end{aligned}$$

where the last inequality follows from Lemma 4.1.

Conversely, last chain of equivalences also shows that any r.i. space whose norm is equivalent to $\|\varphi(t)f^*(t)\|_{\widehat{E}}$ coincides with the interpolation space

$$(L_1, L_\infty)^K_{\widehat{E}^{\widehat{\varphi}}} = \mathcal{F}(\Lambda_\varphi, M_\varphi),$$

and therefore is an ultrasymmetric space. □

Example 4.4. Let $1 \le q < \infty$ and $-\infty < \alpha < \infty$. The Lorentz-Zygmund spaces $L_{\infty,q}(\log L)^\alpha$ and $L^\alpha_{\exp}$ were introduced in [2] as the spaces of all measurable functions on $(0, 1)$ with the norms

$$\begin{aligned}\|f\|_{\infty,q,\alpha} &= \left(\int_0^1 \big((\ln \tfrac{e}{t})^\alpha f^*(t)\big)^q \frac{dt}{t}\right)^{1/q}, \quad \text{if } \alpha + \frac{1}{q} < 0, \\ \|f\|_{\infty,\infty,\alpha} &= \sup_{0<t<1} (\ln \tfrac{e}{t})^\alpha f^*(t), \qquad\qquad \text{if } \alpha < 0,\end{aligned}$$

respectively. In the case $1 \le q < \infty$, the fundamental function of the space is $\varphi(t) = (\ln(e/t))^{\alpha+\frac{1}{q}}$, $0 < t < 1$, which has both extension indices equal to zero, so the results in [21] does not cover it. We write the norm of the space in the form

$$\|f\|_{\infty,q,\alpha} = \left(\int_0^1 \big(\varphi(t)f^*(t)\big)^q \frac{dt}{t\ln(e/t)}\right)^{1/q}.$$

If $\alpha + \frac{1}{q} < 0$, then $\varphi \in \mathcal{P}$ and, by Theorem 4.3, the Lorentz-Zygmund space $L_{\infty,q}(\log L)^\alpha$ is ultrasymmetric, that is, it is an interpolation space between Λ_φ and M_φ. Similarly, if $\alpha < 0$ the space $L^\alpha_{\exp}$ is ultrasymmetric, that is, it is an interpolation space for the couple $(\Lambda_\varphi, M_\varphi)$ with $\varphi(t) = (\ln(e/t))^\alpha$, $0 < t < 1$.

Example 4.5. Similarly the generalized Lorentz-Zygmund spaces $L_{\infty,q,\mathbb{A}}$ defined in [19] are ultrasymmetric.

Let (Ω, μ) denote a totally σ-finite measure space with a non-atomic measure μ and let $\mathbb{A} = (\alpha_0, \alpha_1)$. The generalized Lorentz-Zygmund space $L_{\infty,q,\mathbb{A}}(\Omega, \mu)$ is the set of all μ-measurable functions such that

$$\|f\|_{\infty,q,\mathbb{A}} = \left(\int_0^\infty \big(\ell^{\mathbb{A}}(t) f^*(t)\big)^q \frac{dt}{t} \right)^{1/q} < \infty,$$

where $\ell^{\mathbb{A}}$ is defined in (2). The fundamental function of this space is $\varphi(t) = \ell^{\mathbb{A}+1}(t)$, $0 < t < 1$, which belongs to $\mathcal{P}$ if $\alpha_0 + \frac{1}{q} < 0 < \alpha_1 + \frac{1}{q}$. Hence, by Theorem 4.3, the generalized Lorentz-Zygmund space $L_{\infty,q,\mathbb{A}}(\Omega,\mu)$ is ultrasymmetric for $\alpha_0 + \frac{1}{q} < 0 < \alpha_1 + \frac{1}{q}$.

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